STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	3	-	7	1	0	2

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2023

MATHEMATICS: ANALYSIS AND APPROACHES **24 August 2023**

HIGHER LEVEL **2 hr**

PAPER 2 **0900 – 1100 hrs**

Thursday

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches** formula booklet is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**110** marks]*.
- Number of printed pages = **15.**
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (54 marks)

Answer **all** questions in the **spaces** provided.

1 **[Maximum mark: 4]**

Find the number of 3-digit numbers that can be formed using the digits 1, 2 and 3 when

(a) no repetitions are allowed, [1]

(b) any repetitions are allowed, [1]

(c) each digit may be used at most twice. [2]

TURN OVER

2 **[Maximum mark: 6]**

In the expansion of $\left(x-\frac{1}{2}\right)^n$ for $n \in \mathbb{Z}^+$, the coefficient of x^{n-2} is $\frac{45}{4}$.

(a) Find the value of n. [4]

(b) Hence find the middle term in the expansion. [2]

TURN OVER

3 **[Maximum mark: 5]**

Show that $4\left(\cos^4\theta+\sin^4\theta\right)\equiv 3+\cos 4\theta$.

TURN OVER

4 **[Maximum mark: 7]**

Two fair cubical dice are thrown simultaneously.

Let X_1 and X_2 denote the number shown on die 1 and die 2 respectively.

The random variable Y is the larger number shown on the two dice or the common number shown on the dice if the numbers are equal.

(a) **Complete** the following table which gives the possible outcomes of Y.

x_2 \ x_1	**1**	**2**	**3**	**4**	**5**	**6**
1	1					
2		2				
3			3			
4				4		
5					5	
6						6

[2]

(b) Find $\mathrm{P}(Y = y)$ for $y = 1, 2, \ldots, 6$. [2]

(c) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [3]

TURN OVER

TURN OVER

5 **[Maximum mark: 7]**

Triangle ABC is right-angled at B and the length of the sides are in geometric sequence with common ratio r where $r > 1$.

(a) Find the value of r. [4]

(b) Hence find the possible values of $\tan \angle BAC$. [3]

TURN OVER

6 **[Maximum mark: 5]**

It is given that $\lambda \mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$ where λ is a positive constant, $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors.

(a) State and explain the geometrical relationship between $\mathbf{a}$ and $\mathbf{b}$? [2]

(b) Find the value of $|\mathbf{b}|$ in terms of λ. [3]

TURN OVER

7 [Maximum mark: 9]

A continuous random variable X has probability density function

$$f(x)=\begin{cases}\dfrac{1}{4}x\mathrm{e}^{-\frac{1}{2}x}, & x>0\\ 0, & \text{otherwise.}\end{cases}$$

(a) Find the mode of X. [2]

(b) Show that the mean of X is 4.

[You may assume that $x^2\mathrm{e}^{-\frac{1}{2}x}\to 0$ as $x\to\infty$.] [5]

(c) Find the probability that X lies between the mode and the mean. [2]

TURN OVER

TURN OVER

8 [Maximum mark: 11]

The ceiling function $\lceil x \rceil$ maps a real number x to the least integer greater than or equal to x. For example, $\lceil 2.1 \rceil = 3$ and $\lceil -3.8 \rceil = -3$.

The function f is defined by

$$f(x) = \begin{cases} \lceil x \rceil, & -2 < x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$$

(a) Find the value of $f(-1.4)$. [1]

(b) Sketch the graph of $y = f(x)$ for $-2 < x \le 2$. [2]

(c) Explain why the inverse function f^{-1} does not exist. [2]

(d) Find the range of f. [1]

The function g is defined by $g(x) = \dfrac{3x-3}{x-3}$, $x \in \mathbb{R}, x \ne 3$.

(e) Find $(g \circ g)(x)$. [2]

(f) Hence find the range of $g^{-1} \circ f$. [3]

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TURN OVER

TURN OVER

Do NOT write solutions on this page

SECTION B (56 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

9 [Maximum mark: 24]

A family of cubic functions is defined by $f_k(x) = k^2x^3 - kx^2 + x$, where $k \in \mathbb{Z}^+$.

(a) Find the exact roots of $f_1(x) = 0$. [3]

(b) Show that $f_k(x) = 0$ has only one real root for any $k \in \mathbb{Z}^+$. [4]

(c) Express $f_k'(x)$ and $f_k''(x)$ in terms of k. [2]

(d) Show that $\left(\frac{1}{3k}, \frac{7}{27k}\right)$ is a point of inflexion of $y = f_k(x)$. [5]

(e) Find the equation of the line that joins the points of inflexion of $y = f_1(x)$ and $y = f_2(x)$. [4]

(f) Show that all the points of inflexion of $y = f_k(x)$ for all $k \in \mathbb{Z}^+$ lie on a straight line. [1]

(g) Find

$$\int x\, f_k\left(1-x^2\right)\mathrm{d}x.$$

Leave your answer in terms of k. [5]

TURN OVER

10 [Maximum mark: 12]

The random variable X is normally distributed with mean μ and standard deviation σ.

It is known that $\mathrm{P}(X < m) = 0.2$ and $\mathrm{P}(X < 7) = 0.8$.

(a) Find the value of $\mathrm{P}(m < X < 7)$. [2]

(b) Find the value of $\mathrm{P}(\mu < X < 7)$. [1]

(c) Express μ in terms of m. [2]

(d) Given that $\sigma = \sqrt{5}$, find the value of μ. [2]

Ten independent observations of X are randomly selected.

It is given that $2\mathrm{P}(X < r) = 3\mathrm{P}(X > r)$ for a certain constant r.

(e) (i) Show that $\mathrm{P}(X > r) = 0.4$.

(ii) Find the probability that there are more observations of X with values greater than r than observations of X with values less than r in the selection.

[5]

TURN OVER

11 **[Maximum mark: 20]**

Consider the differential equation

$$(x\ln x)\frac{\mathrm{d}y}{\mathrm{d}x}+y=2\ln x$$

where $y = 0$ when $x = 2$.

(a) Use Euler's method with a step length of 0.1 to find an approximate value of y when $x = 2.5$.

Copy and complete the table below. Leave all answers to **5 significant figures.**

x	2	2.1	2.2	2.3	2.4	2.5
y	0					

[5]

(b) Suggest how the approximation found can be made more accurate. [1]

(c) Show that $\ln x$ is an integrating factor for this differential equation. [4]

(d) Hence by solving the differential equation, show that its solution can be expressed in the form

$$y=\ln x+\frac{c}{\ln x},$$

where c is a constant to be determined. [5]

(e) Find the value of y when $x = 2.5$. [1]

(f) On the same axes, sketch the graph of the function $y = f(x)$ found in **part (d)** and graph the points using the table of values found in **part (a)**.
Hence, explain why the approximations are over-estimates of the actual values.
[4]

End of Paper