2026 J2 H2 FM
Linear Algebra (Set 1) – Solutions

1* **Do not use a calculator in answering this question.**

The matrix **P** is given by $\mathbf{P}=\begin{pmatrix}5 & 3 & -1 & 9\\ 3 & 2 & -1 & 5\\ 1 & 1 & 1 & -1\end{pmatrix}$.

(a) Find a matrix **B** such that $\mathbf{BP}=\begin{pmatrix}1 & 1 & 1 & -1\\ 0 & 1 & a & b\\ 0 & 0 & 1 & c\end{pmatrix}$, where a, b and c are some constants to be determined. [3]

(b) Hence solve the equation $\mathbf{Px}=\begin{pmatrix}-4\\ 1\\ 2\end{pmatrix}$. [3]

(c) The equations below represent three planes

$$5x+3y-z=d$$
$$3x+2y-z=e$$
$$x+y+z=f$$

where d, e and f are constants.
What possible geometrical relationship(s) can you say about the three planes? Justify your answer. [2]

<table>
<tr><td>1)</td><td>NJC/Prelim/2020/02/Q1</td><td></td></tr>
<tr><td></td><td>
Recall “Elementary Matrices”
Consider the following matrix multiplication,
$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 2\\ 1 & 3 & 4\\ 1 & 0 & -1\end{pmatrix}=\begin{pmatrix}1 & 2 & 2\\ 1+k & 3+0k & 4-k\\ 1 & 0 & -1\end{pmatrix}$$
The new matrix obtained is almost similar to the original matrix except that the second row is added by k times of third row. This is similar to the elementary row operation R_2+kR_3.

Thus, performing an elementary row operation on a matrix is equivalent to pre-multiplying a matrix by an elementary matrix that represents the elementary row operation. An elementary matrix is a square matrix obtained from an identity matrix by performing a <u>single</u> elementary row on the identity matrix. Some examples of elementary matrices are as follows:
$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ k & 0 & 1\end{pmatrix},\ \begin{pmatrix}1 & 0 & 0\\ 0 & k & 0\\ 0 & 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$
We can see B as a matrix representing a sequence of elementary row operations that is performed on P, to get $\begin{pmatrix}1 & 1 & 1 & -1\\ 0 & 1 & a & b\\ 0 & 0 & 1 & c\end{pmatrix}$.

We can find B by performing the same sequence of elementary row operations on the identity matrix.
</td><td></td></tr>
</table>

	$\left(\begin{array}{cccc\|ccc} 5 & 3 & -1 & 9 & 1 & 0 & 0 \\ 3 & 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$ $\xrightarrow[R3-3(R1)]{R1\leftrightarrow R3}\left(\begin{array}{cccc\|ccc} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 3 & 2 & -1 & 5 & 0 & 1 & 0 \\ 5 & 3 & -1 & 9 & 1 & 0 & 0 \end{array}\right)$ $\xrightarrow[R3-5R1]{R2-3R1}\left(\begin{array}{cccc\|ccc} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & -4 & 8 & 0 & 1 & -3 \\ 0 & -2 & -6 & 14 & 1 & 0 & -5 \end{array}\right)$ $\xrightarrow{-R2}\left(\begin{array}{cccc\|ccc} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 4 & -8 & 0 & -1 & 3 \\ 0 & -2 & -6 & 14 & 1 & 0 & -5 \end{array}\right)$ $\xrightarrow{R3+2R1}\left(\begin{array}{cccc\|ccc} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 4 & -8 & 0 & -1 & 3 \\ 0 & 0 & 2 & -2 & 1 & -2 & 1 \end{array}\right)$ $\xrightarrow{0.5R3}\left(\begin{array}{cccc\|ccc} 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 4 & -8 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 & 0.5 & -1 & 0.5 \end{array}\right)$ $\therefore \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 0.5 & -1 & 0.5 \end{pmatrix}$ $a = 4,\ b = -8,\ c = -1$	M1: Row operations to get first two rows correct. A1: Matrix B A1: a, b, c values

(b)	$\mathbf{Px} = \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \quad \Rightarrow \mathbf{BPx} = \mathbf{B}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}$ $\Rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & -8 \\ 0 & 0 & 1 & -1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 0.5 & -1 & 0.5 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 4 & -8 & 5 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix} \xrightarrow[R1-R3]{R2-4R3} \begin{pmatrix} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 & 13 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$ $\xrightarrow{R1-R2}\begin{pmatrix} 1 & 0 & 0 & 4 & -9 \\ 0 & 1 & 0 & -4 & 13 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}$	M1 for multiplying by **B** and getting RHS. M1 writing augmented matrix.

<table>
<tr>
<td></td>
<td>$x_1 + 4x_4 = -9,$
$x_2 - 4x_4 = 13,$
$x_3 - x_4 = -2$
Let $x_4 = t, t \in \mathbb{R}$. Thus $\mathbf{x} = \begin{pmatrix} -9-4t \\ 13+4t \\ -2+t \\ t \end{pmatrix}$, $t \in \mathbb{R}$.</td>
<td>A1 for correct answer.</td>
</tr>
<tr>
<td>(c)</td>
<td>
<table>
<tr>
<td>Method 1:
$5x+3y-z=d$
$3x+2y-z=e$
$x+y+z=f$
$\Rightarrow \begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$
From (b),
$\begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Thus $\begin{pmatrix} 5 & 3 & -1 \\ 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ is invertible
and hence the system of linear equations has a unique solution for any constants d, e and f, implying that the three planes have exactly one common point of intersection.</td>
<td>Method 2
Line of intersection for the first two planes is parallel to
$\mathbf{d} = (5\mathbf{i}+3\mathbf{j}-\mathbf{k}) \times (3\mathbf{i}+2\mathbf{j}-\mathbf{k})$
$= -\mathbf{i}+2\mathbf{j}+\mathbf{k}$
$\mathbf{d} \bullet \mathbf{n}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \neq 0$
Line of intersection between 1st and 2nd plane not parallel to 3rd plane.
So, three planes will intersect at a point.</td>
</tr>
</table>
</td>
<td>B1 for stating invertible

B1 for one common point of intersection.</td>
</tr>
</table>

2* The sets A and B are defined as follows:

$$A=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 : x+y-z=0\right\},\ B=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 : 2x+3y-5z=1\right\}.$$

(a) Show that A is a subspace of $\mathbb{R}^3$ but B is not a subspace of $\mathbb{R}^3$. [3]

(b) Determine if $A\cup B'$ is a subspace of $\mathbb{R}^3$. [2]

2	**DHS/Prelim/2017/01/Q1a) and b)i) modified**	
a)	**Method 1: Direct Proof** The zero vector is in A. Let $\mathbf{u},\mathbf{v}\in A$. $(\mathbf{u}+\mathbf{v})\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}=\mathbf{u}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}+\mathbf{v}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}$ $=0+0=0$ Therefore $\mathbf{u}+\mathbf{v}\in A$ and A is closed under vector addition. $(a\mathbf{u})\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}=a\left(\mathbf{u}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}\right)=0$ where a is a scalar. Therefore $a\mathbf{u}\in A$ and A is closed under scalar multiplication. Therefore, A is a subspace of $\mathbb{R}^3$. **Method 2: Using Corollary** Let $\mathbf{u}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix},\mathbf{v}=\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}\in A$, then $x_i+y_i-z_i=0$ for $i=1,2$ $\Rightarrow\alpha(x_1+y_1-z_1)+\beta(x_2+y_2-z_2)=0$ $\therefore\alpha\mathbf{u}+\beta\mathbf{v}\in A$ So, A is a subspace of $\mathbb{R}^3$. The zero vector is not in B since $2(0)+3(0)-5(0)=0\neq1$ Therefore, B is a not a subspace of $\mathbb{R}^3$.	 **B1** **B1** **B1**

(b)	Both $\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\in B'\subseteq A\cup B'$ However, $\begin{pmatrix}0\\2\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix}\notin A$, $\begin{pmatrix}0\\2\\1\end{pmatrix}\in B$ (i.e. $\notin B'$). Therefore $\begin{pmatrix}0\\2\\1\end{pmatrix}\notin A\cup B'$. Since $A\cup B'$ is not closed under vector addition, it is not a subspace of $\mathbb{R}^3$.	**B1** **B1**

3* **(a)** Given that **A** is an invertible (non-singular) 3 x 3 matrix, show that the first column of $\mathbf{A}^{-1}$ is the solution of the equation $\mathbf{Ax}=\begin{pmatrix}1\\0\\0\end{pmatrix}$. [1]

(b) Give the equations whose solutions are the second and third columns of $\mathbf{A}^{-1}$ respectively. [1]

(c) Using parts **(a)** and **(b)**, find the inverse of **P**, where $\mathbf{P}=\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}$ for unknown constants a, b and c. [3]

(d) Using the result in part **(c)**, find the inverse of the matrix **Q** where $\mathbf{Q}=\begin{pmatrix}1&0&0\\a&1&0\\b&c&1\end{pmatrix}$. [1]

3	**ASRJC Promo 9649/2020/Q2**	
(a)	Let $\mathbf{A}^{-1}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ $\mathbf{Ax}=\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow\mathbf{x}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}$ ie the solution is the first column of A^{-1}	**B1**
(b)	**The equations whose solutions are the second and third columns of** A^{-1} are $\mathbf{Ax}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\mathbf{Ax}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively.	**B1**

(c)	By (a) and (b), solutions to $\mathbf{Px}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{Px}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{Px}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ gives the 1st, 2nd and 3rd columns of $\mathbf{P}^{-1}$ respectively. $\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}x_1+ax_2+bx_3\\x_2+cx_3\\x_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow x_3=0, x_2=0, x_1=1$ first column of $\mathbf{P}^{-1}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\begin{pmatrix}x_1+ax_2+bx_3\\x_2+cx_3\\x_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\Rightarrow x_3=0, x_2=1, x_1=-a$ second column of $\mathbf{P}^{-1}=\begin{pmatrix}-a\\1\\0\end{pmatrix}$ $\begin{pmatrix}x_1+ax_2+bx_3\\x_2+cx_3\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\Rightarrow x_3=1, x_2=-c, x_1=ac-b$ third column of $\mathbf{P}^{-1}=\begin{pmatrix}ac-b\\-c\\1\end{pmatrix}$ Therefore $\mathbf{P}^{-1}=\begin{pmatrix}1&-a&ac-b\\0&1&-c\\0&0&1\end{pmatrix}$	**B1** **B1** **B1**
(d)	$\mathbf{Q}=\begin{pmatrix}1&0&0\\a&1&0\\b&c&1\end{pmatrix}=\mathbf{P}^T$ $\mathbf{Q}^{-1}=\left(\mathbf{P}^T\right)^{-1}=\left(\mathbf{P}^{-1}\right)^T=\begin{pmatrix}1&0&0\\-a&1&0\\ac-b&-c&1\end{pmatrix}$	**B1**

4 **Unsupported answers from a graphing calculator are not allowed in this question.**

In response to the government's effort to promote food source diversification and local production, a local entrepreneur set up Apollo Poultry Farm. The farm has a total poultry rearing area of 200 m^2. The entrepreneur plans to rear c chickens, d ducks, g geese and q quails to be sold every three months. The rearing space needed for each chicken, duck, goose and quail is 0.5 m^2, 1.5 m^2, 2 m^2 and 0.5 m^2 respectively. Each chicken, duck, goose and quail consumes 10 kg, 15 kg, 20 kg and 10 kg of poultry feed in three months. The labour cost of caring for each chicken, duck, goose and quail for three months is \$5, \$10, \$15 and \$10 respectively, which will be paid to a farm hand as salary. Every month, the entrepreneur has available 1000 kg of poultry feed and a budget of \$800 to pay the farm hand.

(a) Express the constraints in rearing space, feed and labour cost as inequalities in terms of c, d, g and q. [3]

For the following parts, assume that the resources in rearing space, feed and labour cost are all fully utilised.

(b) Apply elementary row operations to solve the system of linear equations, expressing c, d and g in terms of q. [4]

(c) For every chicken, duck, goose and quail sold, the entrepreneur expects to earn profits of \$5, \$12, \$20 and \$10 respectively. Find the maximum profit that he can make in three months, and state the corresponding values of c, d, g and q. [4]

(d) Based on the answer obtained, what advice would you give to the entrepreneur? [1]

4	**HCI Promo 9649/2020/Q9**	
(a)	$10c+15d+20g+10q \le 3000$ …(1) $5c+10d+15g+10q \le 2400$ …(2) $0.5c+1.5d+2g+0.5q \le 200$ …(3)	**B1** **B1** **B1**
(b)	$\left[\begin{array}{cccc\|c} 10 & 15 & 20 & 10 & 3000 \\ 5 & 10 & 15 & 10 & 2400 \\ 0.5 & 1.5 & 2 & 0.5 & 200 \end{array}\right]$ $R_1 \to R_1 \div 10$: $\left[\begin{array}{cccc\|c} 1 & 1.5 & 2 & 1 & 300 \\ 5 & 10 & 15 & 10 & 2400 \\ 0.5 & 1.5 & 2 & 0.5 & 200 \end{array}\right]$ $R_2 \to R_2 - 5R_1$ $R_3 \to R_3 - 0.5R_1$: $\left[\begin{array}{cccc\|c} 1 & 1.5 & 2 & 1 & 300 \\ 0 & 2.5 & 5 & 5 & 900 \\ 0 & 0.75 & 1 & 0 & 50 \end{array}\right]$ $R_2 \to R_2 \div 2.5$: $\left[\begin{array}{cccc\|c} 1 & 1.5 & 2 & 1 & 300 \\ 0 & 1 & 2 & 2 & 360 \\ 0 & 0.75 & 1 & 0 & 50 \end{array}\right]$ $R_1 \to R_1 - 1.5R_2$	**M1 – first row leading 1**

	$R_3 \to R_3 - 0.75R_2$: $$\left[\begin{array}{cccc\|c} 1 & 0 & -1 & -2 & -240 \\ 0 & 1 & 2 & 2 & 360 \\ 0 & 0 & -0.5 & -1.5 & -220 \end{array}\right]$$ $R_3 \to R_3 \div -0.5$: $$\left[\begin{array}{cccc\|c} 1 & 0 & -1 & -2 & -240 \\ 0 & 1 & 2 & 2 & 360 \\ 0 & 0 & 1 & 3 & 440 \end{array}\right]$$ $R_1 \to R_1 + R_3$ $R_2 \to R_2 - 2R_3$: $$\left[\begin{array}{cccc\|c} 1 & 0 & 0 & 1 & 200 \\ 0 & 1 & 0 & -4 & -520 \\ 0 & 0 & 1 & 3 & 440 \end{array}\right]$$ $c = 200 - q$ $d = -520 + 4q$ $g = 440 - 3q$	**M1 – rref** **A1** **A1**
(c)	$c = 200 - q \geq 0 \Rightarrow q \leq 200$ $d = -520 + 4q \geq 0 \Rightarrow q \geq 130$ $g = 440 - 3q \geq 0 \Rightarrow q \leq 146\frac{2}{3}$ Profit $= 5c + 12d + 20g + 10q$ $= 5(200 - q) + 12(-520 + 4q) + 20(440 - 3q) + 10q$ $= 3560 - 7q$ Maximum profit $= 3560 - 7(130) = \$2650$ $c = 70$, $d = 0$, $g = 50$, $q = 130$	**M1** **M1** **A1** **A1**
(d)	From the answer, we see that ducks do not contribute to the profit. The entrepreneur should concentrate on rearing other poultry instead of ducks.	**B1**