RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 5: Introduction to Complex Numbers

SYLLABUS INCLUDES

Complex numbers expressed in Cartesian form and Argand diagrams

- extension of the number system from real numbers to complex numbers
- complex roots of quadratic equations
- modulus, argument and conjugate of a complex number
- four operations of complex numbers
- equality of complex numbers
- conjugate roots of a polynomial equation with real coefficients
- representation of complex numbers in the Argand diagram
- geometrical effects of conjugation, negation, addition, subtraction, and multiplication by i

PRE-REQUISITES

- Trigonometry
- Coordinate Geometry
- Vectors
- Algebraic manipulation

CONTENT

1 Introduction to the Imaginary Number i

We know that the solution to the equation $x^2+1=0$ cannot be a real number, as the square of a real number cannot be negative. We say $x^2=-1$ has no real roots.

In order to solve the above equation, we need to find a "number" whose square is -1.

Let's suppose such a "number" exists.

Since we imagined it, let's call this number the **imaginary number** i.

We define i as

$$\boxed{\mathrm{i}=\sqrt{-1}}$$

Hence the solutions to $x^2+1=0$ are $x=\pm\sqrt{-1}=\pm\mathrm{i}$.

Example 1

(a) If $\mathrm{i}=\sqrt{-1}$, simplify $\mathrm{i}^2,\ \mathrm{i}^3,\ \mathrm{i}^4,\ \mathrm{i}^{2009},\ \mathrm{i}^{2010},\ \mathrm{i}^{2011},\ \mathrm{i}^{2012}$.

Solution

$\mathrm{i}=\mathrm{i}$ $\quad$ $\mathrm{i}^2=-1$ $\quad$ $\mathrm{i}^3=(\mathrm{i}^2)(\mathrm{i})=-\mathrm{i}$ $\quad$ $\mathrm{i}^4=(\mathrm{i}^2)(\mathrm{i}^2)=1$

$\mathrm{i}^{2009}=(\mathrm{i}^4)^{502}(\mathrm{i})=\mathrm{i}$ $\quad$ $\mathrm{i}^{2010}=(\mathrm{i}^4)^{502}(\mathrm{i}^2)=-1$ $\quad$ $\mathrm{i}^{2011}=(\mathrm{i}^4)^{502}(\mathrm{i}^3)=-\mathrm{i}$ $\quad$ $\mathrm{i}^{2012}=(\mathrm{i}^4)^{503}=1$

Let's generalize: If k is a positive integer, then

$$\mathrm{i}^{4k}=1,\quad \mathrm{i}^{4k+1}=\mathrm{i},\quad \mathrm{i}^{4k+2}=-1,\quad \mathrm{i}^{4k+3}=-\mathrm{i}$$

(b) Perform the four basic operations on i:

(i) $\mathrm{i}+\mathrm{i}=2\mathrm{i}$ $\qquad$ **(ii)** $5\mathrm{i}-\mathrm{i}=4\mathrm{i}$

(iii) $5\mathrm{i}\times3\mathrm{i}=15\mathrm{i}^2=-15$ $\qquad$ **(iv)** $6\mathrm{i}\div3\mathrm{i}=2$

(c) Solve for x if $x^2-2x+2=0$.

Solution

$$x^2-2x+2=0$$

$$x^2-2x+1=-1$$

$$\Leftrightarrow (x-1)^2=-1$$

$$x-1=\pm\mathrm{i}$$

$$x=1\pm\mathrm{i}$$

OR: $x^2-2x+2=0$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}$$

$$=\frac{2\pm\sqrt{-4}}{2}$$

$$=\frac{2\pm2\mathrm{i}}{2}=1\pm\mathrm{i}$$

2 Complex Numbers

Notice that the solution to Example 1(c) is a combination of real numbers and imaginary numbers. Such numbers are called complex numbers.

2.1 Definition of a Complex Number

A **complex number** is of the form $x+\mathrm{i}y$ where x and y are <u>real numbers</u> and $\mathrm{i}=\sqrt{-1}$.

The set of complex numbers is denoted by $\mathbb{C}=\{z : z=x+\mathrm{i}y,\ x,\ y\in\mathbb{R}\}$.

x is known as the <u>real</u> part of z, denoted by $\underline{\mathrm{Re}(z)}$, and

y is known as the <u>imaginary</u> part of z, denoted by $\underline{\mathrm{Im}(z)}$.

Note that $\mathrm{Im}(z)$ **<u>does</u> <u>not</u> include** i.

$x+\mathrm{i}y$ is known as the **cartesian form** of the complex number *z*.

Example 2

Write down the real and imaginary parts of the following complex numbers:

z	**Re(*z*)**	**Im(*z*)**
–2 + 3i	–2	3
1 – i	1	–1
– 4	–4	0
5i	0	5

Remarks:

- $\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}\subseteq\mathbb{C}$
- If $y=0$, then $z=x$ is a real number.
- If $x=0$, then $z=\mathrm{i}y$ is a purely imaginary number.
- Complex numbers cannot be ordered, i.e. given any two complex numbers z_1 and z_2, we cannot compare whether $z_1<z_2$ unless they are real numbers.

2.2 Operations on Complex Numbers

In this section, let $a,\ b,\ c,\ d \in \mathbb{R}$ and $z_1,\ z_2,\ z_3 \in \mathbb{C}$.

(a) Equality of Two Complex Numbers

2 complex numbers are equal if and only if their corresponding real and imaginary parts are equal, i.e. $a + \mathrm{i}b = c + \mathrm{i}d \quad \Leftrightarrow \quad a = c$ and $b = d$.

For example, if $x, y \in \mathbb{R}$ and $x + \mathrm{i}y = 5 - 3\mathrm{i}$, we have $\underline{x = 5 \text{ and } y = -3}$

(b) Addition of Complex Numbers

$$(a + \mathrm{i}b) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d)$$

Addition of complex numbers is commutative: $z_1 + z_2 = z_2 + z_1$
Addition of complex numbers is associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

For example, $(3 + 4\mathrm{i}) + (1 - \mathrm{i}) = \underline{4 + 3\mathrm{i}}$

(c) Subtraction of Complex Numbers

$$(a + \mathrm{i}b) - (c + \mathrm{i}d) = (a - c) + \mathrm{i}(b - d)$$

For example, $(3 + 4\mathrm{i}) - (1 - \mathrm{i}) = \underline{2 + 5\mathrm{i}}$

(d) Multiplication of Complex Numbers

$$\begin{aligned}(a + \mathrm{i}b)(c + \mathrm{i}d) &= ac + \mathrm{i}ad + \mathrm{i}bc + \mathrm{i}^2 bd = ac + \mathrm{i}ad + \mathrm{i}bc + (-1)bd \\ &= (ac - bd) + \mathrm{i}(ad + bc)\end{aligned}$$

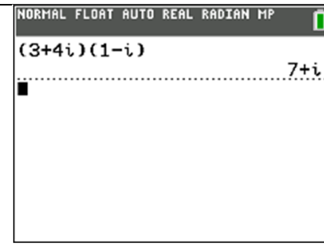
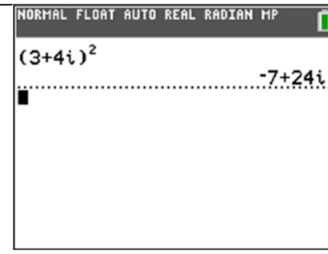
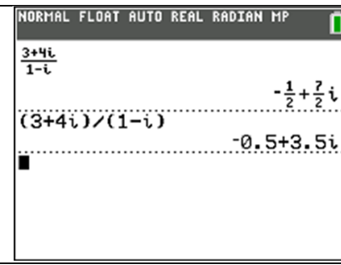
Multiplication of complex numbers is commutative: $z_1 z_2 = z_2 z_1$
Multiplication of complex numbers is associative: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
Multiplication of complex numbers is distributive over addition: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

For example, $(3 + 4\mathrm{i})(1 - \mathrm{i}) = \underline{3 - 3\mathrm{i} + 4\mathrm{i} - 4\mathrm{i}^2 = 7 + \mathrm{i}}$ (note that $\mathrm{i}^2 = -1$)

$(3 + 4\mathrm{i})^2 = (3 + 4\mathrm{i})(3 + 4\mathrm{i}) \ \underline{= 9 + 12\mathrm{i} + 12\mathrm{i} + 16\mathrm{i}^2 = -7 + 24\mathrm{i}}$

Remarks: The above manipulations of complex numbers are the same as algebraic manipulations of expressions such as $(a + b)(c + d), (a + b) \pm (c + d)$ and so on. The only additional consideration is $\mathrm{i}^2 = -1$.

We can use the GC to perform operations on complex numbers

Operation	Example	GC Screen
Multiplication of complex numbers (Press $\boxed{\text{2nd}}$ **.** to get i)	$(3+4\mathrm{i})(1-\mathrm{i})$	
Square of a complex number	$(3+4\mathrm{i})^2$	
Division of complex numbers	$\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$	

Question: How did the GC obtain $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}} = -\dfrac{1}{2}+\dfrac{7}{2}\mathrm{i}$?

(e) Division of Complex Numbers

Recall when we tried to simplify $\dfrac{2-\sqrt{3}}{1+\sqrt{2}}$, we multiply it by $\dfrac{1-\sqrt{2}}{1-\sqrt{2}}$ so that we could rationalize the denominator.

For $\dfrac{3+4\mathrm{i}}{1-\mathrm{i}}$, we will multiply it by $\dfrac{1+\mathrm{i}}{1+\mathrm{i}}$, so that

$$\frac{3+4\mathrm{i}}{1-\mathrm{i}} \underline{= \frac{3+4\mathrm{i}}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{3+3\mathrm{i}+4\mathrm{i}+4\mathrm{i}^2}{1+1} = \frac{-1+7\mathrm{i}}{2} = -\frac{1}{2}+\frac{7}{2}\mathrm{i}}$$

In general, $\dfrac{a+\mathrm{i}b}{c+\mathrm{i}d} = \dfrac{a+\mathrm{i}b}{c+\mathrm{i}d} \times \dfrac{c-\mathrm{i}d}{c-\mathrm{i}d} = \dfrac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2-\mathrm{i}^2d^2} = \dfrac{(a+\mathrm{i}b)(c-\mathrm{i}d)}{c^2+d^2}$

Remark: Note that $c-\mathrm{i}d$ is chosen based on the denominator of $\dfrac{a+\mathrm{i}b}{c+\mathrm{i}d}$.

We call $c-\mathrm{i}d$, the complex conjugate of $c+\mathrm{i}d$.

Example 3 [RJC Prelim 9233/2005/01/Q1(i)]

The complex numbers z and w are such that $z=-1+2\mathrm{i}$ and $w=1+b\mathrm{i}$, where $b\in\mathbb{R}$.
Given that the imaginary part of $\dfrac{w}{z}$ is $-\dfrac{3}{5}$, find the value of b.

Solution

$$\frac{w}{z}=\frac{1+b\mathrm{i}}{-1+2\mathrm{i}}\times\frac{-1-2\mathrm{i}}{-1-2\mathrm{i}}=\frac{-1-2\mathrm{i}-b\mathrm{i}-2b\mathrm{i}^2}{(-1)^2+2^2}=\frac{-1+2b}{5}+\mathrm{i}\frac{(-2-b)}{5}$$

Given $\mathrm{Im}\left(\dfrac{w}{z}\right)=-\dfrac{3}{5} \Rightarrow \dfrac{-2-b}{5}=-\dfrac{3}{5} \Rightarrow b=1$

Example 4

Find the square roots of 24 – 10i.
By completing the square, or otherwise, solve $z^2-6z=15-10\mathrm{i}$.

Solution:
Let the square roots of 24 – 10i be $x+y\mathrm{i}$, where $x,y\in\mathbb{R}$.

Then
$$\begin{aligned}24-10\mathrm{i}&=(x+y\mathrm{i})^2\\&=x^2+2xy\mathrm{i}+y^2\mathrm{i}^2\\&=x^2-y^2+2xy\mathrm{i}\end{aligned}$$

Comparing real and imaginary parts:

$$24=x^2-y^2 \quad ---(1)$$
$$-10=2xy \quad ---(2)$$

From (2):
$$y=-\frac{5}{x} \quad ---(3)$$

Subst (3) in (1):
$$24=x^2-\frac{25}{x^2}$$

$$x^4-24x^2-25=0$$
$$(x^2-25)(x^2+1)=0$$
$$x^2=25 \quad \text{or} \quad x^2=-1 \quad \text{(N.A since } x\in\mathbb{R})$$
$$x=\pm 5$$

When $x=5,\ y=-1$

When $x=-5,\ y=1$

$\therefore$ the square roots are $5-\mathrm{i}$ or $-5+\mathrm{i}$.

Note that evaluating $\sqrt{24-10\mathrm{i}}$ using GC only gives us $5-\mathrm{i}$.

$$
\begin{aligned}
z^2 - 6z = 15 - 10\mathrm{i} \quad &\Rightarrow \quad (z-3)^2 - 3^2 = 15 - 10\mathrm{i} \\
&\Rightarrow \quad (z-3)^2 = 24 - 10\mathrm{i} \\
&\Rightarrow \quad z - 3 = \pm(5 - \mathrm{i}) \quad \text{(from above result)} \\
&\Rightarrow \quad z = 8 - \mathrm{i} \ \text{ or } \ z = -2 + \mathrm{i}
\end{aligned}
$$

Note:

We can use GC to check our answers.

Substitute $z = 8 - \mathrm{i}$ and $z = -2 + \mathrm{i}$ into $z^2 - 6z$, and check that both give $15-10\mathrm{i}$ as answers.

2.3 Complex Conjugates

The **complex conjugate** of $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, is denoted by z^* and defined as

$$z^* = x - \mathrm{i}y$$

Observe that $\mathrm{Re}(z^*) = x = \mathrm{Re}(z)$ and $\mathrm{Im}(z^*) = -y = -\mathrm{Im}(z)$.
Note that $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$ are conjugates of each other, and we call them a conjugate pair.

2.4 Some Properties of Complex Conjugates

The following properties can be easily proven by letting $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ where $x, y, u, v \in \mathbb{R}$.

Properties	**Proofs**	**Example**
(a) $(z^*)^* = z$	$[(x+\mathrm{i}y)^*]^* = (x-\mathrm{i}y)^* = x+\mathrm{i}y$	$[(1-3\mathrm{i})^*]^*$ $=(1+3\mathrm{i})^*$ $=1-3\mathrm{i}$
(b) $z + z^* = 2\,\mathrm{Re}(z)$	$(x+\mathrm{i}y)+(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $=2x$	$(1-3\mathrm{i})+(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})+(1+3\mathrm{i})$ $=2$
(c) $z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	$(x+\mathrm{i}y)-(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $=2\mathrm{i}y$	$(1-3\mathrm{i})-(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})-(1+3\mathrm{i})$ $=-6\mathrm{i}$

(d)	$zz^* = x^2 + y^2$	$(x+\mathrm{i}y)(x+\mathrm{i}y)^*$ $=(x+\mathrm{i}y)(x-\mathrm{i}y)$ $=x^2-(\mathrm{i}y)^2$ (difference of 2 squares) $=x^2-\mathrm{i}^2y^2$ ($\mathrm{i}^2=-1$) $=x^2+y^2$	$(1-3\mathrm{i})(1-3\mathrm{i})^*$ $=(1-3\mathrm{i})(1+3\mathrm{i})$ $=1-(3\mathrm{i})^2$ $=1+9$ $=10$
(e)	$z=z^* \Leftrightarrow z\in\mathbb{R}$	$z=z^*$ $\Leftrightarrow x+\mathrm{i}y=x-\mathrm{i}y$ $\Leftrightarrow 2\mathrm{i}y=0$ $\Leftrightarrow \mathrm{Im}(z)=0$ $\Leftrightarrow z$ is real	
(f)	$(z+w)^*=z^*+w^*$	$(x+\mathrm{i}y+u+\mathrm{i}v)^* = [x+u+\mathrm{i}(y+v)]^*$ $=x+u-\mathrm{i}(y+v)$ $=(x-\mathrm{i}y)+(u-\mathrm{i}v)$ $=z^*+w^*$	
(g)	$(zw)^*=z^*w^*$	$[(x+\mathrm{i}y)(u+\mathrm{i}v)]^*=[xu-yv+\mathrm{i}(xv+yu)]^*$ $=xu-yv-\mathrm{i}(xv+yu)$ $(x+\mathrm{i}y)^*(u+\mathrm{i}v)^* = (x-\mathrm{i}y)(u-\mathrm{i}v)$ $=xu-yv-\mathrm{i}(xv+yu)$	$[(1+3\mathrm{i})(1-2\mathrm{i})]^*$ $=(1+\mathrm{i}-6\mathrm{i}^2)^*$ $=(7+\mathrm{i})^*=7-\mathrm{i}$ $(1+3\mathrm{i})^*(1-2\mathrm{i})^*$ $=(1-3\mathrm{i})(1+2\mathrm{i})$ $=1-\mathrm{i}-6\mathrm{i}^2$ $=7-\mathrm{i}$

Note:

We can use **(g)** to show that $(z^2)^*=(z^*)^2$ by letting $w = z$.

We can also show that $(z^n)^*=(z^*)^n$, $n\in\mathbb{Z}^+$

Example 5

Let $z = 1 + \mathrm{i}a$ and $w = 1 + \mathrm{i}b$, where $a,\ b \in \mathbb{R}$ and $a > 0$. If $zw^* = 3 - 4\mathrm{i}$, find the exact values of a and b.

Solution

$$zw^* = 3 - 4\mathrm{i} \quad \Rightarrow \quad (1+\mathrm{i}a)(1-\mathrm{i}b) = 3 - 4\mathrm{i}$$
$$\Rightarrow \quad (1+ab) + \mathrm{i}(a-b) = 3 - 4\mathrm{i}$$

Equating real and imaginary parts, $1 + ab = 3$ ----- (1)

$a - b = -4 \ \Rightarrow\ b = a + 4$ ----- (2)

Subst (2) into (1), $1 + a(a+4) = 3$

$$a^2 + 4a - 2 = 0$$
$$a = \frac{-4 \pm \sqrt{16+8}}{2} = \frac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$$

Since $a > 0$, $a = -2 + \sqrt{6}$ and $b = a + 4 = 2 + \sqrt{6}$

Example 6

Solve the simultaneous equations $z^* + w = -1,\ \ 2z + (\mathrm{i}w)^* = -1$.

Solution

$$z^* + w = -1 \qquad \Rightarrow z + w^* = -1 \text{ ------- } (1)$$
$$2z + (\mathrm{i}w)^* = -1 \qquad \Rightarrow 2z - \mathrm{i}w^* = -1 \text{ ----- } (2)$$

$(1)\times\mathrm{i}$: $\quad \mathrm{i}z + \mathrm{i}w^* = -\mathrm{i}$ ----- (3)

$(3)+(2)$: $\quad (\mathrm{i}+2)z = -\mathrm{i} - 1$

$$z = \frac{-\mathrm{i}-1}{\mathrm{i}+2} = -\frac{3}{5} - \frac{1}{5}\mathrm{i}$$

From (1),

$$w^* = -1 - z = -\frac{2}{5} + \frac{1}{5}\mathrm{i}$$
$$w = -\frac{2}{5} - \frac{1}{5}\mathrm{i}$$

3 Roots of Polynomials

With complex numbers, we have the following theorem.

Fundamental Theorem of Algebra:

A polynomial of degree n has n roots (real or non-real).

Thus, taking non-real roots into account, a quadratic polynomial (degree 2) always has 2 roots, a cubic polynomial (degree 3) always has 3 roots, and so on.

Furthermore, if all the coefficients of the polynomial are real, we have the following result:

Non-real roots of a polynomial with <u>real coefficients</u> occur in conjugate pairs.

In other words, if β is a non-real root of a polynomial with <u>real coefficients</u>, then β^* is also a non-real root.

Note that real coefficients include the constant term as well.

[Refer to Appendix for the proof of this result]

From **Example 1(c)** we obtained $x = 1 \pm \mathrm{i}$ as conjugate pair solutions to $x^2 - 2x + 2 = 0$.

Example 7 (Quadratic)
Find the roots of the equation $z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$.

Solution

$$z^2 + (-1+4\mathrm{i})z + (-5+\mathrm{i}) = 0$$

$$z = \frac{-(-1+4\mathrm{i}) \pm \sqrt{(-1+4\mathrm{i})^2 - 4(-5+\mathrm{i})}}{2}$$ [$(-1+4\mathrm{i})^2$ can be evaluated using GC]

$$= \frac{(1-4\mathrm{i}) \pm \sqrt{5-12\mathrm{i}}}{2}$$ [$\sqrt{5-12\mathrm{i}}$ can be evaluated using GC]

$$= \frac{(1-4\mathrm{i}) \pm (3-2\mathrm{i})}{2}$$

$$= 2-3\mathrm{i} \text{ or } -1-\mathrm{i}$$

Question: Why are the roots not in conjugate pairs?

Answer: Because not all the coefficients of the quadratic polynomial are real.

Example 8 (Cubic) Do not use a calculator in answering this question.

Find the exact roots of the equation $z^3-2z^2+2z-1=0$.

Solution

Since all the coefficients of the cubic polynomial are real, it has either 3 real roots or 1 real root and 1 conjugate pair of non-real roots.

$z=1$ is a solution of $z^3-2z^2+2z-1=0$ since $1-2+2-1=0$. So $z-1$ is a factor of z^3-2z^2+2z-1.

Observe that (or via long division) $z^3-2z^2+2z-1=(z-1)(z^2-z+1)$.

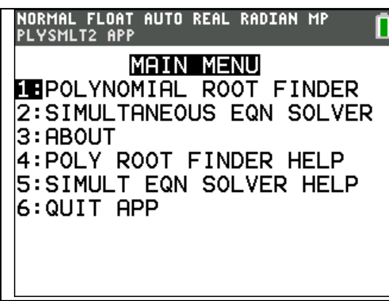
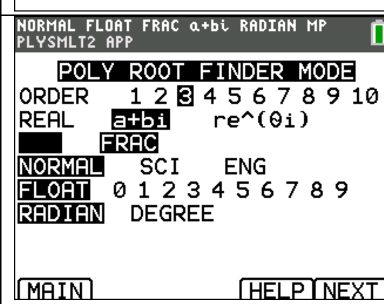
Hence $z^3-2z^2+2z-1=0 \Rightarrow (z-1)(z^2-z+1)=0$

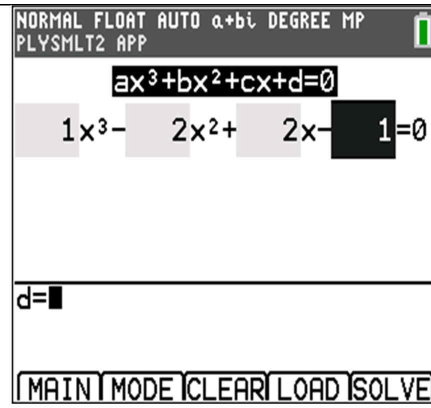
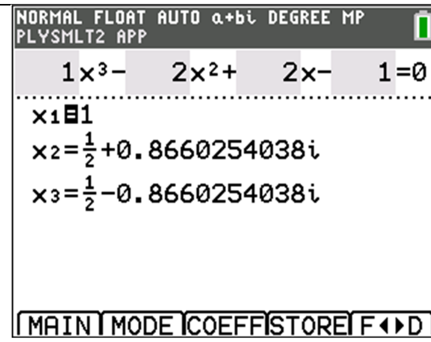
$$\Rightarrow z-1=0 \text{ or } z^2-z+1=0$$

$$\Rightarrow z=1 \text{ or } z=\frac{1\pm\sqrt{1-4}}{2}$$

$$\text{i.e. } z=1 \text{ or } z=\frac{1+\mathrm{i}\sqrt{3}}{2} \text{ or } z=\frac{1-\mathrm{i}\sqrt{3}}{2}$$

Note that the **Polynomial Root Finder** of the **PlySmlt2** app in the GC can be used to solve equations with real coefficients.

1. Press **[APPS]** and select **PlySmlt2**. 2. Select **1: Polynomial Root Finder**.	
3. Adjust the settings as depicted on the screen to solve a cubic equation. Remember to select "a+bi" or "re^(θi)" for the GC to display all roots, not just the real roots. 4. Press **[GRAPH]** which is the button below **NEXT**.	

5. Key in the values of the coefficients a_3, a_2, a_1 and a_0. 6. Press **[GRAPH]** to solve the system of equations.	
7. Read off the answers. x_1, x_2 and x_3 give the 3 roots of z. Note : The GC cannot display the roots in exact form.	

Example 9 (Cubic) Do not use a calculator in answering this question.
Given that $1-\mathrm{i}$ is a root of the equation $z^3-5z^2+8z+p=0$, where $p\in\mathbb{R}$, find the other 2 roots and the value of p.

Solution
Method 1:
Since all the coefficients of the cubic polynomial are real, non-real roots will occur in conjugate pairs. Given $1-\mathrm{i}$ is a root, then $1+\mathrm{i}$ is also a root.

The third root must be a real number k.

$$\begin{aligned} z^3-5z^2+8z+p &= [z-(1+\mathrm{i})]\,[z-(1-\mathrm{i})](z-k) \\ &= [(z-1)-\mathrm{i}]\,[(z-1)+\mathrm{i}](z-k) \\ &= [(z-1)^2-\mathrm{i}^2](z-k) \\ &= (z^2-2z+2)(z-k) \end{aligned}$$

Comparing coefficient of z, $8=2k+2 \;\Rightarrow\; k=3$

Comparing the constant term, $p=-2k=-6$

The other 2 roots are $1+\mathrm{i}$, 3, and $p=-6$.

Method 2:

Since $1-\mathrm{i}$ is a root of the given equation, an alternative way of finding p is by substituting the root into the equation.

$$z^3-5z^2+8z+p=0$$
$$(1-\mathrm{i})^3-5(1-\mathrm{i})^2+8(1-\mathrm{i})+p=0$$
$$(-2-2\mathrm{i})-5(-2\mathrm{i})+8(1-\mathrm{i})+p=0$$
$$p=-6$$

since $(1-\mathrm{i})^2=1-2\mathrm{i}+\mathrm{i}^2=-2\mathrm{i}$

$(1-\mathrm{i})^3=(1-\mathrm{i})^2(1-\mathrm{i})=-2\mathrm{i}(1-\mathrm{i})=-2-2\mathrm{i}$

Show these working since the calculator is not used for this question.

Since all the coefficients of the cubic polynomial are real, non-real roots will occur in conjugate pairs. Given $1-\mathrm{i}$ is a root, then $1+\mathrm{i}$ is also a root.

The third root must be a real number k.

$$z^3-5z^2+8z-6=[z-(1+\mathrm{i})]\,[z-(1-\mathrm{i})](z-k)$$
$$=(z^2-2z+2)(z-k)$$

Comparing the constant term, $-6=-2k \Rightarrow k=3$

The other 2 roots are $1+\mathrm{i}$ and 3.

Example 10 (Quartic) Do not use a calculator in answering this question.

One root of the equation $2x^4+8x^3+x^2+ax+b=0$, where a and b are real, is $x=-2+\mathrm{i}$. Find the values of a and b, and the other roots.

Solution

Since $a,\ b\in\mathbb{R}$, then all the coefficients of the polynomial are real, so non-real roots will occur in conjugate pairs. Since $-2+\mathrm{i}$ is a root, so $-2-\mathrm{i}$ is also a root. A quadratic factor of the quartic polynomial is

$$\left[x-(-2+\mathrm{i})\right]\left[x-(-2-\mathrm{i})\right]=\left[(x+2)-\mathrm{i}\right]\left[(x+2)+\mathrm{i}\right]$$
$$=(x+2)^2-\mathrm{i}^2=x^2+4x+5$$

Hence $2x^4+8x^3+x^2+ax+b=\left(x^2+4x+5\right)\left(2x^2+cx+d\right)$

Note that the 2nd quadratic factor is $2x^2+cx+d$, not x^2+cx+d, since the quartic polynomial has leading coefficient 2.

Compare coefficients of

$$x^3:\quad 8=c+8 \quad\Rightarrow\quad c=0$$
$$x^2:\quad 1=d+0+10 \quad\Rightarrow\quad d=-9$$
$$x:\quad a=4d+0=-36$$
$$x^0:\quad b=5d=-45$$

Now the quartic polynomial can be written as

$$2x^4+8x^3+x^2-36x-45=\left(x^2+4x+5\right)\left(2x^2-9\right)$$
$$=\left(x^2+4x+5\right)\left(\sqrt{2}x-3\right)\left(\sqrt{2}x+3\right)$$

Hence the other roots of the equation are $-2-\mathrm{i}$, $\dfrac{3}{\sqrt{2}}$ and $\dfrac{-3}{\sqrt{2}}$.

Example 11 (Quadratic with non-real coefficients)
Do not use a calculator in answering this question.

Given that $-1+2\mathrm{i}$ is a root of the equation $3z^2-(2+9\mathrm{i})z-11+7\mathrm{i}=0$, find the second root of the equation in cartesian form, showing your working.

Solution

Note that not all the coefficients of the polynomial are real, so we cannot say that non-real roots occur in conjugate pairs. Let the 2nd root of the equation be w. Then

$$\begin{aligned}3z^2-(2+9\mathrm{i})z-11+7\mathrm{i}&=3\left[z-(-1+2\mathrm{i})\right]\left[z-w\right]\\&=3\left[z^2-(-1+2\mathrm{i}+w)z+(-1+2\mathrm{i})w\right]\end{aligned}$$

Comparing coefficients of z, $\quad -(2+9\mathrm{i})=-3(-1+2\mathrm{i}+w)$

$$w=\left(\frac{2}{3}+3\mathrm{i}\right)+1-2\mathrm{i}=\frac{5}{3}+\mathrm{i}$$

Self-Practice

Useful properties involving Conjugates

Consider a complex number $z=x+\mathrm{i}y,\ x,\ y\in\mathbb{R}$, and its conjugate $z^*=x-\mathrm{i}y$, then

(i)	$z=z^* \quad\Leftrightarrow\quad z$ is real	**(v)**	$(z\pm w)^*=z^*\pm w^*$
(ii)	$z+z^*=2x=2\mathrm{Re}(z)$	**(vi)**	$(zw)^*=z^*w^*$
(iii)	$z-z^*=2\mathrm{i}y=2\mathrm{Im}(z)\ \mathrm{i}$	**(vii)**	$(z^n)^*=(z^*)^n$, where $n\in\mathbb{Z}^+$
(iv)	$zz^*=\left\lvert z^2\right\rvert$	**(viii)**	$\left(\dfrac{z}{w}\right)^*=\dfrac{z^*}{w^*}$

4 Geometrical Representation of a Complex Number

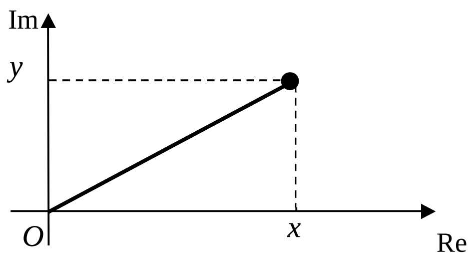
Recall that we represent real number on a number line.
For complex numbers, since there are imaginary numbers, it makes sense for us to have an imaginary number line as well.

4.1 Argand Diagram

A complex number $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$ can be represented by the point $P(x, y)$. This idea was formally introduced by the French mathematician Jean-Robert Argand, and hence the diagram which represents a complex number in this way was named after him.

An **Argand diagram** is similar to the Cartesian plane with the

- Horizontal axis (labeled as Re) representing the real part of z, and
- Vertical axis (labeled as Im) representing the imaginary part of z.



Remarks:

Points on the horizontal axis represent real numbers and points on the vertical axis represent purely imaginary numbers.

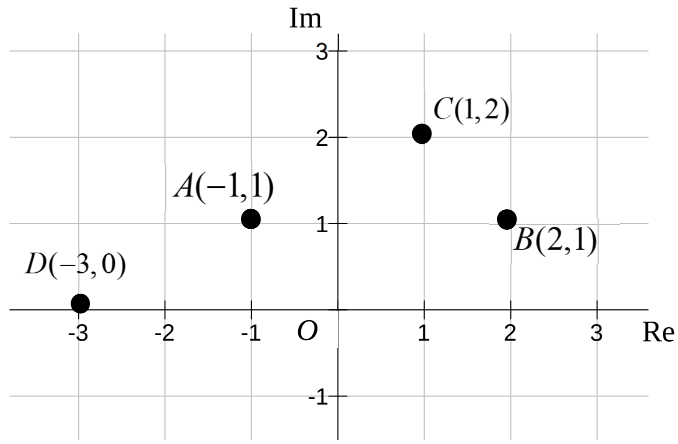
On an Argand diagram, complex numbers behave (in terms of additions and subtractions) like 2-D vectors, but they are NOT vectors. In particular, we can divide complex numbers, but we cannot do the same to vectors.

4.2 Geometrical Representation of Addition and Subtraction of Complex Numbers

Consider the complex numbers $z_1 = -1 + \mathrm{i}$ and $z_2 = 2 + \mathrm{i}$.

Then $z_1 + z_2 = 1 + 2\mathrm{i}$
and $z_1 - z_2 = -3$.

Let the points A, B, C and D represent z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ respectively on an Argand diagram.

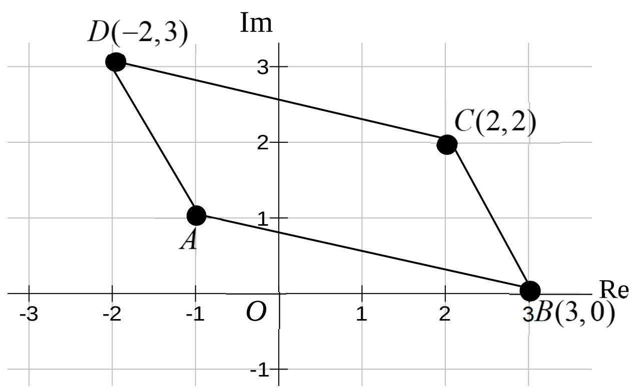


Note that $OACB$ is a parallelogram.
As mentioned, the complex numbers behave like vectors on an Argand diagram.
(Recall how we proved $OACB$ is a parallelogram using vectors.)

Example 12

The points A, B, C and D represent four complex numbers z_1, $z_2 = 3$, $z_3 = 2 + 2\mathrm{i}$ and $z_4 = -2 + 3\mathrm{i}$ respectively. Given that $ABCD$ forms a parallelogram in an anti-clockwise sense, find z_1 by calculation (i.e. not by drawing).

Solution



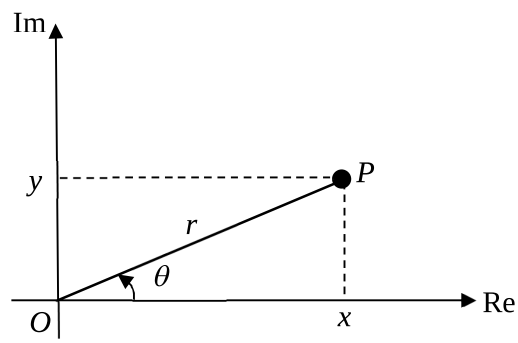
$z_1 - z_4 = z_2 - z_3$
$\Rightarrow z_1 = z_2 - z_3 + z_4$
$= 3 - (2 + 2\mathrm{i}) + (-2 + 3\mathrm{i})$
$= -1 + \mathrm{i}$

Recall that if this is a vector question:
Since $ABCD$ is a parallelogram,
$\overrightarrow{DA} = \overrightarrow{CB}$
$\overrightarrow{OA} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OC}$
$\overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC} + \overrightarrow{OD}$

4.3 Modulus and Argument of a Complex Number

Let the point P represent the complex number $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$ on an Argand diagram.
We are interested in 2 geometrical measurements:

- $r =$ <u>distance</u> of P from the origin O
- $\theta =$ <u>directed angle</u> that $\overrightarrow{OP}$ makes with the <u>positive real axis</u>
  ($\theta > 0$ when measured in an anti–clockwise sense;
  $\theta < 0$ when measured in a clockwise sense)



r is known as the **modulus** of z and is denoted by $|z|$, where

$$|z| = r = \sqrt{x^2 + y^2} .$$

Clearly $r \geq 0$.

θ is known as the (principal) **argument** of z and is denoted by $\arg(z)$, where

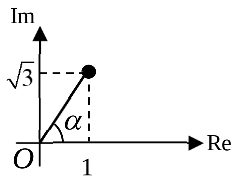
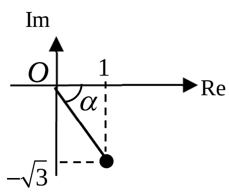
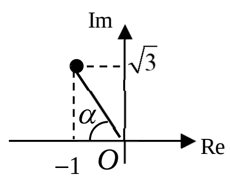
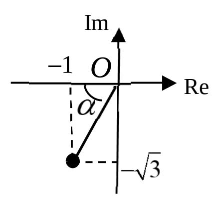
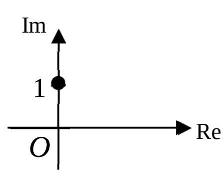
$$-\pi < \arg(z) \leq \pi .$$

$\arg(z)$ should be given in radians.

Example 13

For each of the following complex number, find the exact value of the modulus and argument:

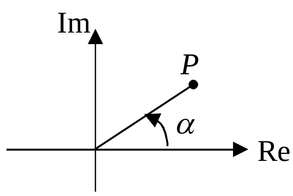
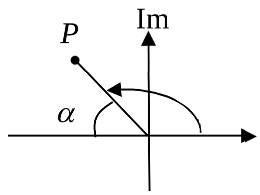
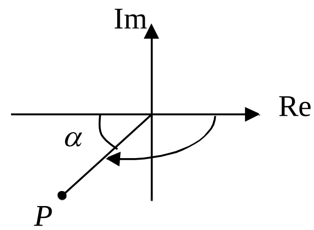
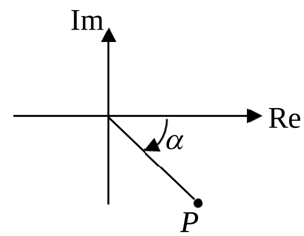
(i) $1+\mathrm{i}\sqrt{3}$ **(ii)** $1-\mathrm{i}\sqrt{3}$ **(iii)** $-1+\mathrm{i}\sqrt{3}$ **(iv)** $-1-\mathrm{i}\sqrt{3}$ **(v)** i

<table>
<tr><td>Solution
(i)

</td><td>$\left|1+\mathrm{i}\sqrt{3}\right| = \sqrt{1+3} = 2$
$\tan\alpha = \sqrt{3} \;\Rightarrow\; \alpha = \frac{\pi}{3} \;\Rightarrow\; \arg(1+\mathrm{i}\sqrt{3}) = \alpha = \frac{\pi}{3}$</td></tr>
<tr><td>(ii)

</td><td>$\left|1-\mathrm{i}\sqrt{3}\right| = \sqrt{1+3} = 2$
$\tan\alpha = \sqrt{3} \;\Rightarrow\; \alpha = \frac{\pi}{3} \;\Rightarrow\; \arg(1-\mathrm{i}\sqrt{3}) = -\alpha = -\frac{\pi}{3}$</td></tr>
<tr><td>(iii)

</td><td>$\left|-1+\mathrm{i}\sqrt{3}\right| = \sqrt{1+3} = 2$
$\tan\alpha = \sqrt{3} \;\Rightarrow\; \alpha = \frac{\pi}{3} \;\Rightarrow\; \arg(-1+\mathrm{i}\sqrt{3}) = \pi - \alpha = \frac{2\pi}{3}$</td></tr>
<tr><td>(iv)

</td><td>$\left|-1-\mathrm{i}\sqrt{3}\right| = \sqrt{1+3} = 2$
$\tan\alpha = \sqrt{3} \;\Rightarrow\; \alpha = \frac{\pi}{3}$
$\Rightarrow\; \arg(-1-\mathrm{i}\sqrt{3}) = -(\pi - \alpha) = -\frac{2\pi}{3}$</td></tr>
<tr><td>(v)

</td><td>$\left|\mathrm{i}\right| = 1$
$\arg(\mathrm{i}) = \frac{\pi}{2}$</td></tr>
</table>

<u>Finding arg(z):</u>

Consider a point P on an Argand diagram representing $z = x + \mathrm{i}y,\ x,\ y \in \mathbb{R}$

Always first identify the quadrant that z lies, then let **basic angle**, $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$.

1^{st} **quadrant**	2^{nd} **quadrant**	3^{rd} **quadrant**	4^{th} **quadrant**
			
arg(z) = α	arg(z) = $\pi - \alpha$	arg(z) = $-(\pi - \alpha)$	arg(z) = $-\alpha$

Questions

1. What is $\arg(z)$ if z is real?
2. What is $\arg(z)$ if z is purely imaginary?
3. What is $|z|$ and $\arg(z)$ if $z = 0$?

Answers

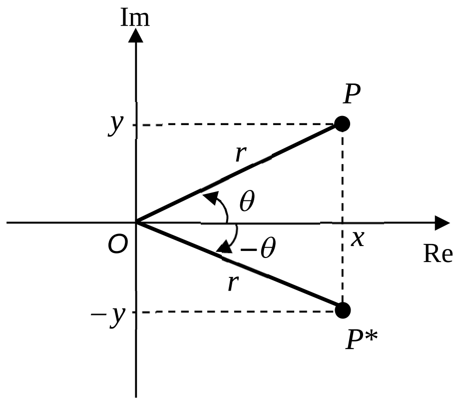
1. If z is real, $\arg(z) = \begin{cases} 0, & \text{for } z > 0, \\ \pi, & \text{for } z < 0. \end{cases}$

2. If z is purely imaginary, $\arg(z) = \begin{cases} \dfrac{\pi}{2}, & \text{for } \mathrm{Im}(z) > 0, \\ -\dfrac{\pi}{2}, & \text{for } \mathrm{Im}(z) < 0. \end{cases}$

3. If $z = 0$, $|z| = 0$ but $\arg(z)$ is not defined.

4.4 Geometrical Effects of Conjugation and Negation

Let the points P and P^* represent the complex numbers $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, $x, y \in \mathbb{R}$ respectively on an Argand diagram.

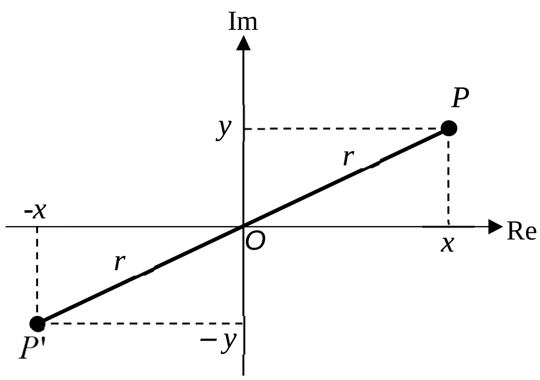


Observe that P and P^* are reflections of each other about the real axis.

From the diagram, we see that

$$|z^*| = |z| \text{ and } \arg(z^*) = -\arg z .$$

Let the points P and P' represent the complex numbers $z = x + \mathrm{i}y$ and $-z = -x - \mathrm{i}y$, $x, y \in \mathbb{R}$ respectively on an Argand diagram.



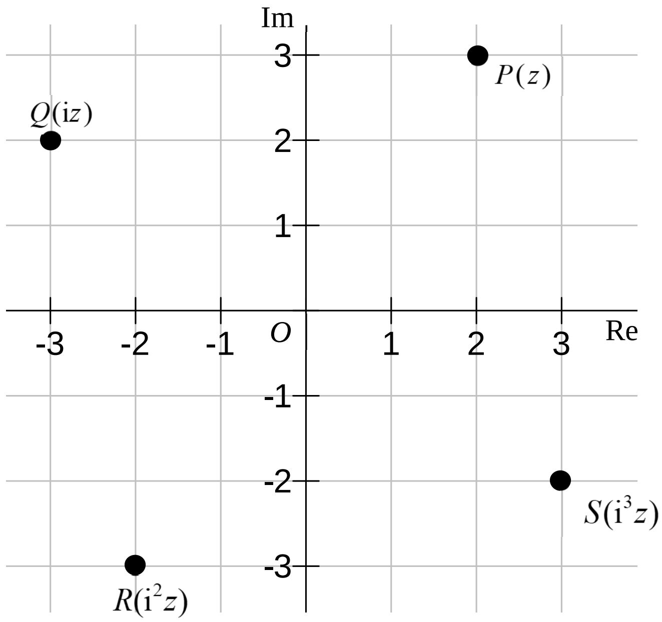
Observe that P and P' are reflections of each other about the origin so POP' is a straight line. We can also say that P' is obtained by rotating P 180° about the origin.

Note that $|-z| = |z|$ while

$\arg z$ and $\arg(-z)$ differ by π .

4.5 Geometrical Effect of Multiplying a Complex Number by i

Consider the complex number $z = 2 + 3\mathrm{i}$.
Draw the points P, Q, R and S representing z, $\mathrm{i}z$, $\mathrm{i}^2 z$ and $\mathrm{i}^3 z$ respectively on the same Argand diagram.



Now $\mathrm{i}z = -3 + 2\mathrm{i}$, $\mathrm{i}^2 z = -2 - 3\mathrm{i}$, $\mathrm{i}^3 z = 3 - 2\mathrm{i}$

Notice that if *P* is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain *Q*.

If *Q* is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain *R*.

If *R* is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain *S*.

If *S* is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain *P*.

Generally, for a complex number $z = a + \mathrm{i}b$, multiplying it by i results in $\mathrm{i}z = -b + \mathrm{i}a$.

Thus, the geometrical effect of multiplying a complex number z by i is an anti-clockwise rotation of the point representing z through an angle of $\frac{\pi}{2}$ radians about O. Note that the modulus is still the same.

Example 14

In an Argand diagram, the points A, B and C represent the complex numbers a, $6+8\text{i}$ and c respectively. $OABC$ is a square described in an anti-clockwise sense, where O is the origin. Give a reason why $c=\text{i}a$ and a reason why $a+c=6+8\text{i}$. Find a and c by calculation (i.e. not by geometry).

Solution

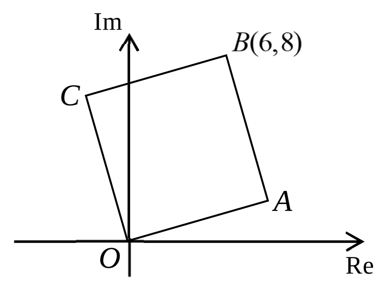
When A is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we get C.

Hence $c=\text{i}a$ ----- (1)

OC is parallel and equal in length to AB, thus

$$c-0=6+8\text{i}-a$$

Thus, $a+c=6+8\text{i}$ ----- (2)



Subst (1) into (2),

$$a+\text{i}a=6+8\text{i}$$

$$a(1+\text{i})=6+8\text{i}$$

$$a=\frac{6+8\text{i}}{1+\text{i}}=7+\text{i}$$

Subst into (1), $c=\text{i}(7+\text{i})=-1+7\text{i}$

4.6 Modulus and Argument of zw and $\dfrac{z}{w}$

Example 15

Let $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ where a, b, c and d are real numbers.

(i) Show that $|zw| = |z||w|$.

(ii) It is further given that a, b, c and d are positive, and $\arg(z) = \theta$, $\arg(w) = \phi$. By considering $\tan(\theta + \phi)$, show that $\arg(zw) = \arg(z) + \arg(w)$.

If $|z| = 2\sqrt{2}$, $\arg z = \dfrac{\pi}{4}$ and $|w| = 2$, $\arg w = \dfrac{\pi}{3}$, find the modulus and argument of zw.

Solution

(i)
$$
\begin{aligned}
|zw| &= |(a+\mathrm{i}b)(c+\mathrm{i}d)| \\
&= |ac - bd + \mathrm{i}(ad+bc)| \qquad \cdots (*) \\
&= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\
&= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2c^2)} \\
&= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} \\
&= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)} \\
&= \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{a^2+b^2}\sqrt{c^2+d^2} = |z||w|
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\tan(\theta+\phi) &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \\
&= \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a}\cdot\frac{d}{c}} \\
&= \frac{\frac{bc+ad}{ac}}{\frac{ac-bd}{ac}} \\
&= \frac{ad+bc}{ac-bd} = \tan(\arg zw) \quad \text{since } zw = ac - bd + \mathrm{i}(ad+bc) \text{ from } (*)
\end{aligned}
$$

Since $zw = ac - bd + \mathrm{i}(ad+bc)$ with a, b, c and d positive, we have $ad + bc > 0$, so $0 < \arg(zw) < \pi$. Note also that $0 < \theta + \phi < \pi$ since $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$.

Since $\tan(\theta+\phi) = \tan(\arg zw)$ and the tangent function is one-one in the interval $(0, \pi)$, we can conclude that $\theta + \phi = \arg zw$, i.e. $\arg(z) + \arg(w) = \arg(zw)$.

Using the results in (i) and (ii),

$$|zw| = |z||w| = 2\sqrt{2}\cdot 2 = 4\sqrt{2}$$

$$\arg(zw) = \arg(z) + \arg(w) = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$$

It can be shown that

(1) $|zw| = |z||w|$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

(2) $\arg(zw) = \arg(z) + \arg(w)$

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

Note that results (1) are the same as those for the real numbers, while results (2) are like the properties of logarithmic functions. For the A-level syllabus, you are not expected to know and recall the above results, however you could be asked to show the result algebraically then use it in a question, as seen in the previous example.

CONCLUSION

Basically, we have added another dimension to the real number system so that we can evaluate the square root of negative numbers. Complex numbers are represented geometrically on an Argand diagram. In the Argand diagram, complex numbers behave like vectors, but complex numbers are different from vectors, because vectors cannot be divided or raised to a power, but complex numbers can.

Complex numbers have essential concrete applications in a variety of sciences and related areas such as signal processing (creation of fractals), control theory, electromagnetism, quantum mechanics, cartography, vibration analysis, fluid dynamics, and many others.

If you are interested in the history of imaginary number i, you may want to refer to the book "An Imaginary Tale, the Story of $\sqrt{-1}$", by Paul J.Nahin, Princeton University Press.

APPENDIX

PROOF OF RESULT THAT NON-REAL ROOTS OF A POLYNOMIAL WITH REAL COEFFICIENTS OCCUR IN CONJUGATE PAIRS

Consider the equation

$$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + ... + a_1 z + a_0 = 0,$$

where $a_n,\ a_{n-1},\ a_{n-2}, ...,\ a_1,\ a_0 \in \mathbb{R},\ a_n \neq 0,\ n \in \mathbb{Z}^+$.

Suppose β is a non-real root of the polynomial,

i.e. $$a_n \beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + ... + a_1\beta + a_0 = 0.$$

Taking conjugates on both sides of the equation,

$$(a_n \beta^n + a_{n-1}\beta^{n-1} + a_{n-2}\beta^{n-2} + ... + a_1\beta + a_0)* = 0*$$

$$(a_n \beta^n)* + (a_{n-1}\beta^{n-1})* + (a_{n-2}\beta^{n-2})* + ... + (a_1\beta)* + a_0* = 0*$$

Now $(\beta^k)* = (\beta*)^k$ and $(a_k)* = a_k$ since $a_k \in \mathbb{R}$.

Clearly $0* = 0$ since $0 \in \mathbb{R}$.

Thus we have

$$a_n(\beta*)^n + a_{n-1}(\beta*)^{n-1} + a_{n-2}(\beta*)^{n-2} + ... + a_1(\beta*) + a_0 = 0,$$

i.e. $\beta*$ is also a non-real root of the given polynomial.

SUMMARY