ANDERSON SERANGOON JUNIOR COLLEGE
JC2 Preliminary Examinations 2025
Higher 3

MATHEMATICS **9820/01**

Paper 1 **2 September 2025**

3 hours

Additional Materials: Answer Booklets
List of Formulae (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

Answer **all** the questions.

1 By considering a suitable scalar product, prove that

$$ax + by + cz^{\ 2} \leq\ a^2 + b^2 + c^2 \quad x^2 + y^2 + z^2$$

for any real numbers *a, b, c, x, y* and *z*. Deduce a necessary and sufficient condition on *a, b, c, x, y* and *z* for the following equation to hold:

$$ax + by + cz^{\ 2} =\ a^2 + b^2 + c^2 \quad x^2 + y^2 + z^2 \qquad [3]$$

(i) Show that $\dfrac{\left(a^2+b^2+c^2\right)^2}{3} \geq (ab)^2 + (bc)^2 + (ca)^2$ for all real numbers *a, b* and *c*. [3]

(ii) Find real numbers *p, q* and *r* that satisfy both

$$p^2 + 4q^2 + r^2 = 729 \quad \text{and} \quad 8p + 8q + r = 243 . \qquad [5]$$

2 The numbers $\mathrm{f}(r)$ satisfy $\mathrm{f}(r) > \mathrm{f}(r+1)$ for $r = 1, 2, 3, \ldots\ldots$ Show that, for any non-negative integer *n*,

$$k^n\left(k-1\right)\mathrm{f}\left(k^{n+1}\right) \leq \sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leq k^n\left(k-1\right)\mathrm{f}\left(k^n\right)$$

where *k* is an integer greater than 1. [3]

(i) By taking $\mathrm{f}(r) = \dfrac{1}{r}$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1$$

Deduce that the sum $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge. [5]

(ii) Let $S(n)$ be the set of positive integers less than *n* which do not contain the digit 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \dfrac{1}{3} + \dfrac{1}{4}$.
Show that $S(1000)$ contains 728 distinct numbers.

Show that $\sigma(n) < 80$ for all *n*. [5]

3 A train has n seats, where $n \geq 2$. For a particular journey, all n seats have been sold and each of the n passengers has been allocated a seat.

The passengers arrive one at a time and are labelled $T_1, T_2, \ldots\ldots, T_n$ according to the order in which they arrive. T_1 arrives first and T_n arrives last. The seat allocated to T_r $r = 1, 2, \ldots\ldots, n$ is labelled S_r.

Passenger T_1 ignores his allocation and decides to choose a seat at random (each of the n seats being equally likely). However, for each $r \geq 2$, passenger T_r sits in S_r, if it is available or, if S_r is not available, chooses from the available seats at random.

(i) Let P_n be the probability that, in a train with n seats, T_n sits in S_n. Write down the value of P_2 and find the value of P_3. [2]

(ii) Explain why, for $k = 2, \ldots\ldots, n-1$,

$$P\ T_n \text{ sits in } S_n \mid T_1 \text{ sits in } S_k \ = P_{n-k+1},$$

and deduce that, for $n \geq 3$,

$$P_n = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right).$$

[6]

(iii) Find P_4. Make a conjecture on the value of P_n in its simplest form and prove your result by induction. [5]

4 Fermat's Little Theorem states that if p is prime and $\gcd(x, p) = 1$, then $x^{p-1} \equiv 1 \mod p$.

Let n and x be positive integers with $\gcd(x, n) = 1$. Show that if $n = mp$ where p is prime and $\gcd(m, p) = 1$, then

$$x^{n-1} \equiv 1 \mod p \quad \text{if and only if } x^{m-1} \equiv 1 \mod p \ .$$ [3]

Chinese Reminder Theorem states that a system

$$\begin{aligned} x &\equiv 1 \mod p_1 \\ x &\equiv 1 \mod p_2 \\ &\ldots\ldots \\ x &\equiv 1 \mod p_k \end{aligned}$$

where $p_1, p_2, \ldots, p_k$ are all pairwise coprime, has unique solution $x \equiv 1 \mod p_1 p_2 \ldots p_k$.

If n is a product of distinct primes. Show that $x^{n-1} \equiv 1 \mod n$ if and only if , for every prime divisor p of n,

$$x^{\left(\frac{n}{p}\right)-1} \equiv 1 \mod p \ .$$ [4]

Deduce that if every prime divisor p of n satisfies $p - 1 \mid n - 1$, then for every x with $\gcd(x, n) = 1$, the congruence

$$x^{n-1} \equiv 1 \mod n$$

holds. [3]

5 **(i)** Let S_n be the number of ways to pick n squares from an $n \times n$ matrix such that no two squares are from the same row and no two squares are from the same column. Find a recurrence relation for S_n. [1]

(ii) Determine a bijection between the ways to pick n squares from an $n \times n$ matrix such that no two squares are from the same row and no two squares are from the same column and the permutation of the set of integers $N = \{1, 2, 3, \ldots, n\}$. Hence state the value of S_n in terms of n. [2]

Let D_n be the number of ways to pick n squares from the same $n \times n$ matrix such that no two squares are from the same row and no two squares are from the same column and no diagonal element is picked. (see Figure 1).

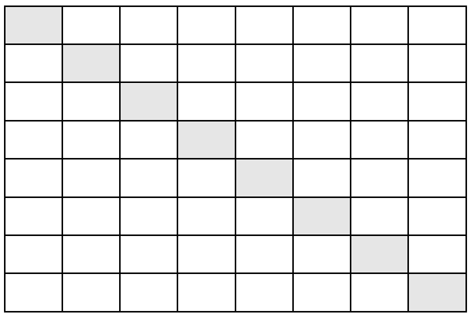

Figure 1. The diagonal elements in the case of an 8×8 matrix

(iii) Show that $D_n = sD_{n-1} + tD_{n-2}$ where s and t are functions of n to be determined. State the values of D_1 and D_2. [4]

(iv) By considering the Principle of inclusion-exclusion, show that D_n can also be expressed as

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$ [4]

(v) A teacher received a different christmas gift from each of her six friends. She conceived the idea of giving a gift in return to each of her friends using these 6 different gifts received. She just need to ensure she doesn't give to each of her friends, the same gift she received from them. How many ways can she do this? [2]

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Investigating the Generalisations of the Product Rule

Throughout this question fg denotes the product of the functions f and g, so that

$$\mathrm{fg}(x) = \mathrm{f}(x) \times \mathrm{g}(x).$$

The differential operator, D, when applied to a function, returns its derivative.
So $\mathrm{Df} = \mathrm{f}'$ and the product rule for the derivative of the product of the functions f and g becomes

$$\mathrm{D}\left(\mathrm{fg}\right) = \mathrm{f\,Dg} + \mathrm{g\,Df}. \qquad (*)$$

The product rule may be generalised in at least two different ways.

Firstly, it can be generalised to give a formula for finding higher derivatives of the product of two functions.

For example, differentiating both sides of (*) and applying the product rule to each term on the right-hand side gives an expression for the second derivative of fg

$$\mathrm{D}^2\left(\mathrm{fg}\right) = \left(\mathrm{f\,D^2g} + \mathrm{Df\,Dg}\right) + \left(\mathrm{Dg\,Df} + \mathrm{g\,D^2f}\right) = \mathrm{f\,D^2g} + 2\,\mathrm{Df\,Dg} + \mathrm{g\,D^2f}$$

A second generalisation of the product rule for two functions gives a formula for the derivative of a product of three or more functions.

For example, if h is a third function, then

$$\mathrm{D}\left(\mathrm{fgh}\right) = \mathrm{D}\left(\mathrm{f}\left(\mathrm{gh}\right)\right) = \mathrm{f\,D}\left(\mathrm{gh}\right) + \mathrm{gh\,Df}.$$

You can then apply the two-term product rule to find the derivative of D(gh) so that

$$\mathrm{D}\left(\mathrm{fgh}\right) = \mathrm{f}\left(\mathrm{g\,Dh} + \mathrm{h\,Dg}\right) + \mathrm{gh\,Df} = \mathrm{fg\,Dh} + \mathrm{fh\,Dg} + \mathrm{gh\,Df}$$

This method generalises so that if $\mathrm{f}_1, \mathrm{f}_2, \ldots\ldots, \mathrm{f}_n$ are functions, then

$$\mathrm{D}\left(\mathrm{f}_1\mathrm{f}_2 \cdots \mathrm{f}_n\right) = \sum_{k=1}^{n} \mathrm{f}_1\mathrm{f}_2 \cdots \mathrm{f}_{k-1}\mathrm{f}_{k+1} \cdots \mathrm{f}_n\,\mathrm{Df}_k.$$

6 **Answer the following questions**

In this question you will investigate these generalisations of the product rule further.

(a) Find an expression for $\mathrm{D}^3(\mathrm{fg})$, the third derivative of the product fg, showing your workings clearly. [2]

The general form of $\mathrm{D}^n(\mathrm{fg})$, the n^{th} derivative of the product fg is as follows:

For $n \in \mathbb{N}$, $\mathrm{D}^n(\mathrm{fg}) = \sum_{i=0}^{n} \binom{n}{i} \left(\mathrm{D}^i \mathrm{f}\right)\left(\mathrm{D}^{n-i} \mathrm{g}\right)$ where $\mathrm{D}^0 \mathrm{f} = \mathrm{f}$.

(b) (i) Show that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$ for positive integers *k* and *i* satisfying $k \geq i$. [2]

(ii) Prove the result for general form of $\mathrm{D}^n(\mathrm{fg})$, the n^{th} derivative of the product fg. [9]

The cubic polynomial $\mathrm{p}(x)$ has three distinct roots r_1, r_2 and r_3.

(c) (i) Find an expression for $\mathrm{D}(\mathrm{p}(x))$. [2]

(ii) Prove that the tangent to the curve $y = \mathrm{p}(x)$ at the midpoint of any two of the roots intersects the curve at the third root. [5]

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