ZHONGHUA SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2022
SECONDARY 4 EXPRESS/ 5 NORMAL (ACADEMIC)

Candidate's Name	Class	Register Number
MARKING SCHEME		

ADDITIONAL MATHEMATICS 4049/02

PAPER 2

13 September 2022
2 hour 15 minutes

Candidates answer on the Question Paper

READ THESE INSTRUCTIONS FIRST

Write your name, class and register number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **90**.

For Examiner's Use
90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Delta = \frac{1}{2}bc\sin A$$

1 **(a)** By using suitable substitution or otherwise, solve the equation $e^{-x}\left(3-e^{x}\right)=5e^{x}$, giving your answer correct to 2 decimal places. [4]

$\dfrac{3}{e^x}-1=5e^x$

$3-e^x=5e^{2x}$

$5e^{2x}+e^x-3=0$ B1- Quadratic equation

Let $u=e^x$

$5u^2+u-3=0$

$u=\dfrac{-1\pm\sqrt{1^2-4(5)(-3)}}{10}$ M1- Quadratic Formula

$=\dfrac{-1\pm\sqrt{61}}{10}$

$e^x=\dfrac{-1+\sqrt{61}}{10}$ (rej. negative)

$x=\ln\left(\dfrac{-1+\sqrt{61}}{10}\right)$ M1

$=-0.384156$

$=-0.38$ (2 d.p) A1

(b) Solve $\sqrt{x+7} = x-5$. [3]

$\sqrt{x+7} = x-5$

$x+7 = (x-5)^2$ M1- Square both sides

$x+7 = x^2 - 10x + 25$

$x^2 - 11x + 18 = 0$

$(x-9)(x-2) = 0$ M1- Solving quadratic equation

$x = 9$ or $x = 2$ (rejected)

$\therefore x = 9$ A1

2 Given that $\tan\theta = p$ and that θ is acute, find in terms of *p*,

(a) $\sin(-\theta)$ [1]

$\sin(-\theta) = -\sin\theta$

$= -\dfrac{p}{\sqrt{1+p^2}}$ B1- o.e

(b) $\tan\left(\dfrac{\pi}{2} - \theta\right)$ [1]

$\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{p}$ B1

3 **(a)** Express $-3x^2-36x-106$ in the form $a(x+b)^2+c$. [2]

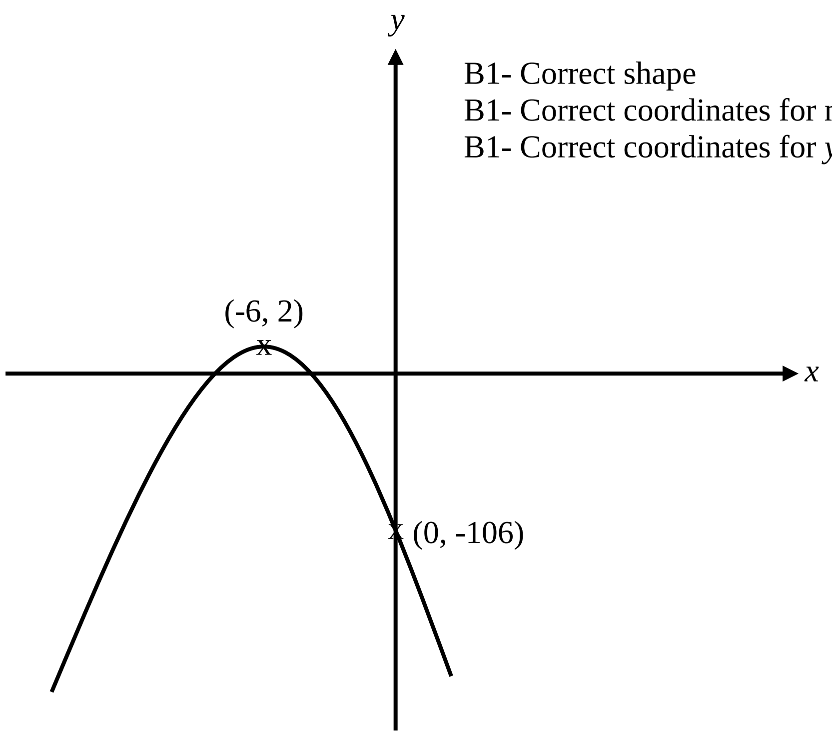
$-3x^2-36x-106$

$=-3\left(x^2+12x+\frac{106}{3}\right)$ M1- Taking out common factor

$=-3\left[(x+6)^2-(6)^2+\frac{106}{3}\right]$

$=-3(x+6)^2+2$ A1

(b) Hence, sketch the graph of $y=-3x^2-36x-106$ on the axes below.
Indicate clearly the coordinates of the points where the graph crosses the y-axis and the turning point on the curve. [3]



B1- Correct shape
B1- Correct coordinates for maximum point
B1- Correct coordinates for y-intercept

4 **(a)** Given that $y = \frac{2+\sin x}{\cos x}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+2\sin x}{\cos^2 x}$. [3]

$$\frac{d}{dx}\left(\frac{2+\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - (-\sin x)(2+\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + 2\sin x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1+2\sin x}{\cos^2 x} \quad \text{(shown)}$$

M1- Differentiating $\cos x$ to get $-\sin x$
M1- Quotient/Product rule

A1- Correct simplification leading to answer

(b) Hence, evaluate $\int_0^{\frac{\pi}{4}} \frac{2+2\sin x}{\cos^2 x}\,\mathrm{d}x$ in exact form. [4]

$$\int_0^{\frac{\pi}{4}} \frac{1+2\sin x}{\cos^2 x}dx = \left[\frac{2+\sin x}{\cos x}\right]_0^{\frac{\pi}{4}}$$

$$= \left(1+2\sqrt{2}\right) - 2$$

$$= 2\sqrt{2} - 1$$

M1- Showing anti-derivative

$$\int_0^{\frac{\pi}{4}} \frac{2+2\sin x}{\cos^2 x}dx = \int_0^{\frac{\pi}{4}} \frac{1+2\sin x}{\cos^2 x}dx + \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2 x}dx$$

B1- Splitting $\frac{2+2\sin x}{\cos^2 x}$

$$= \left(2\sqrt{2}-1\right) + \int_0^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= \left(2\sqrt{2}-1\right) + \left[\tan x\right]_0^{\frac{\pi}{4}}$$

A1- Integrate $\sec^2 x$ to get $\tan x$

$$= \left(2\sqrt{2}-1\right) + \left[1-0\right]$$

$$= 2\sqrt{2}$$

A1

(c) Find the y-coordinate of the maximum turning point of $y = \dfrac{2+\sin x}{\cos x}$ when $0 < x < 2\pi$. [6]

$\dfrac{1+2\sin x}{\cos^2 x} = 0$ M1- $\dfrac{dy}{dx} = 0$

$1+2\sin x = 0$

$\sin x = -\dfrac{1}{2}$ M1- Finding basic angle

Basic angle $= \dfrac{\pi}{6}$

$x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ M1- Values of x

	$\dfrac{7\pi}{6}^-$	$\dfrac{7\pi}{6}$	$\dfrac{7\pi}{6}^+$
$\dfrac{dy}{dx}$	positive	0	negative
Shape			

It is a maximum point at $x = \dfrac{7\pi}{6}$. M1- Use of 1st or 2nd derivative test to determine the maximum point

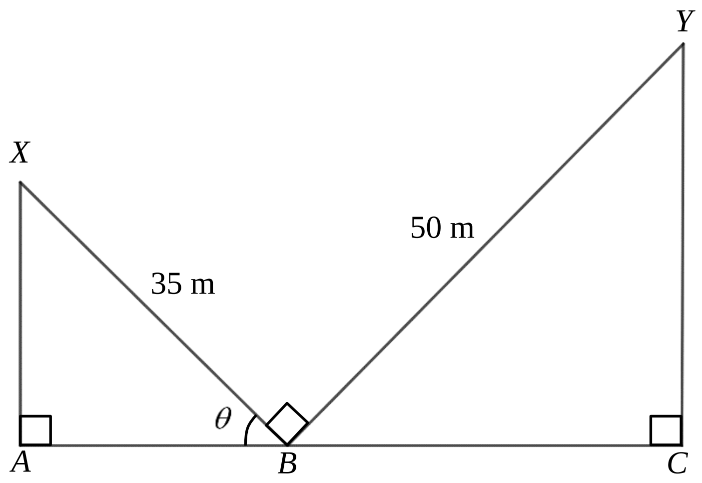
	$\dfrac{11\pi}{6}^-$	$\dfrac{11\pi}{6}$	$\dfrac{11\pi}{6}^+$
$\dfrac{dy}{dx}$	positive	0	negative
Shape			

It is a minimum point at $x = \dfrac{11\pi}{6}$. M1- Use of 1st or 2nd derivative test to determine the min point

y-coordinate of max. turning point $= \dfrac{2+\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}}$

$= -\sqrt{3}$ A1

5



The diagram shows a straight coastline *ABC*. A turtle, at point *X*, at sea, swam 35 m in a straight line to point *B*, where John was throwing food to feed the turtles.
The turtle then turned $90°$ and swam 50 m in a straight line to point *Y* at sea.
Angle $XBY = 90°$, angle $XBA = \theta$ and *XA* and *YC* are perpendicular to *ABC*.

(a) Show that $AC = 35\cos\theta + 50\sin\theta$. [2]

$\cos\theta = \dfrac{AB}{XB}$

$AB = 35\cos\theta$

$\angle YBC = 90° - \theta$

$\angle BYC = \theta$

B1- trigonometric ratios
$\angle BYC = \theta$ must be seen

$\sin\theta = \dfrac{BC}{BY}$

$BC = 50\sin\theta$

B1- Finding *AB* and *BC*

$AC = AB + BC$

$= 35\cos\theta + 50\sin\theta$ (shown) AG

(b) Express AC in the form of $R\cos(\theta-\alpha)$, where $R>0$ and $0°<\alpha<90°$. [4]

$AC = 35\cos\theta + 50\sin\theta$

$R = \sqrt{35^2+50^2}$ M1

$= \sqrt{3725}$

$= 5\sqrt{149}$

$\alpha = \tan^{-1}\left(\frac{50}{35}\right)$ M1

$= 55.00797°$

$= 55.0°$ B1

$\therefore AC = 5\sqrt{149}\cos(\theta-55.0°)$ A1- Accept $\sqrt{3725}$

Do not accept 61.0 or $55.007°$

(c) John's father claims that the coastline AC is 70 m. Without measuring, John said that it was incorrect. Explain, with clear workings, how John arrived at this conclusion. [1]

Max. value of $AC = 5\sqrt{149}$

$= 61.0327$

$= 61.0$ m < 70 m B1- Comparison needs to be seen

$\therefore AC$ cannot be 70 m. Accept equating and showing that there is no solution

6 **(i)** Write down the general term in the binomial expansion of $\left(x-\frac{p}{x^3}\right)^{12}$, where p is a constant. [1]

$$T_{n+1}=\binom{12}{r}(x)^{12-r}\left(-\frac{p}{x^3}\right)^r \quad \text{B1}$$
$$=\binom{12}{r}x^{12-r}(-1)^r(p)^r(x)^{-3r}$$
$$=\binom{12}{r}x^{12-4r}(-1)^r(p)^r$$

(ii) Write down the power of x in this general term. [1]

Power of $x=12-4r$ B1

(iii) The constant term in the binomial expansion of $\left(x-\frac{p}{x^3}\right)^{12}$ is -27500. Find the value of p. [3]

$$12-4r=0$$
$$r=3 \quad \text{B1}$$

$$\text{Constant term}=\binom{12}{3}(-p)^3$$

$$\binom{12}{3}(-p)^3=-27500 \quad \text{M1- equate to -27500}$$
$$220(-p)^3=-27500$$
$$(-p)^3=-125$$
$$p^3=125$$
$$p=5 \quad \text{A1}$$

(iv) Hence, show that every term in the expansion of $\left(1+\frac{x^{12}}{525}\right)\left(x-\frac{p}{x^3}\right)^{12}$ is dependent on x. [4]

$\left(1+\frac{x^{12}}{525}\right)\left(x-\frac{5}{x^3}\right)^{12}$

Coefficient of x^{-12} in $\left(x-\frac{5}{x^3}\right)^{12}$:

$12-4r=-12$	B1
$-4r=-24$	
$r=6$	
$\binom{12}{6}x^{-12}\left(-5\right)^{6}=14437500x^{-12}$	M1- Finding coefficient of x^{-12}
Constant term $=\left(1\right)\left(-27500\right)+\left(\frac{1}{525}\right)\left(14437500\right)$	M1- Finding the constant term (need to see both terms)
$=0$	
$\therefore$ Every term is dependent on x.	A1

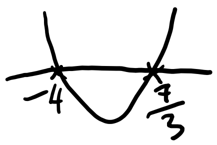
7 **(a)** The equation of a curve is $y = 3x^2 + 5x - 16$.

Find the range of values for which the curve lies below the line $y = 12$. [3]

$3x^2 + 5x - 16 < 12$ B1- Forming correct inequality

$3x^2 + 5x - 28 < 0$

$(3x-7)(x+4) < 0$ M1- Factorisation



$\therefore -4 < x < \dfrac{7}{3}$ A1

(b) Given that the line $y = 3x + k$ is a tangent to the curve $y^2 = bx$, where k and b are positive constants, show that b is a multiple of k. [3]

$(3x+k)^2 = bx$ M1- Substitution

$9x^2 + 6kx + k^2 = bx$

$9x^2 + (6k-b)x + k^2 = 0$

Discriminant $= 0$

$(6k-b)^2 - 4(9)(k^2) = 0$ M1- Equating discriminant to 0, with substitution of values

$36k^2 - 12bk + b^2 - 36k^2 = 0$

$-12bk + b^2 = 0$

$b^2 = 12bk$

$b = 12k$ (shown) A1- Ignore if never reject $b = 0$

8 **(a)** A formula used in a science experiment is $ax^2 + by^2 = 1$, where a and b are constants. Values of x for different values of y have been collected. Explain how a straight line graph can be drawn to represent the formula, and state how the values of a and b could be obtained from the line. [4]

Plot y^2 against x^2 (or x^2 against y^2).

B1- Accept "draw" but to remind students to write "plot"

$$ax^2 + by^2 = 1 \qquad\qquad ax^2 + by^2 = 1$$

$$by^2 = -ax^2 + 1 \qquad \text{or} \qquad ax^2 = -by^2 + 1$$

$$y^2 = -\frac{a}{b}x^2 + \frac{1}{b} \qquad\qquad x^2 = -\frac{b}{a}y^2 + \frac{1}{a}$$

M1- Manipulating into the form Y= mX + c

Where the gradient is $-\frac{a}{b}$ (or $-\frac{b}{a}$) and the vertical intercept is $\frac{1}{b}$ (or $\frac{1}{a}$) respectively.

B1- Gradient

B1- Vertical-intercept

Accept:

$$b = \frac{1}{\text{vertical intercept}}$$

$$a = -\frac{1}{\text{vertical intercept}} \times \text{gradient}$$

(b) A particle, moving in a certain medium with speed v m/s, experiences a resistance to motion of R N (Newton). It is known that R and v are related by the equation $R = kv^a$, where k and a are constants. The table below shows experimental values of the variables v and R.

$\boldsymbol{v}$	5	10	15	20	25	30
$\boldsymbol{R}$	32	96	180	290	410	550

(i) On the grid on the next page, plot $\lg R$ against $\lg v$ and draw a straight line graph.
Hence, estimate the value for each of the constants k and a. [6]

lg *v*	0.6989	1	1.1760	1.3010	1.3979	1.4771
lg *R*	1.5051	1.9822	2.2552	2.4623	2.6127	2.7403

$R = kv^a$

$\lg R = \lg k + a \lg v$

$\lg R = a \lg v + \lg k$

Graph
B1- Correct points
B1- Straight line drawn

Gradient $= \dfrac{2.7 - 1.75}{1.45 - 0.85}$ (from graph)

M1- Gradient from graph

$= \dfrac{19}{12}$

$a = 1.58$ (3 s.f)

A1- Value of a

Vertical intercept $= 0.4$

$\lg k = 0.4$

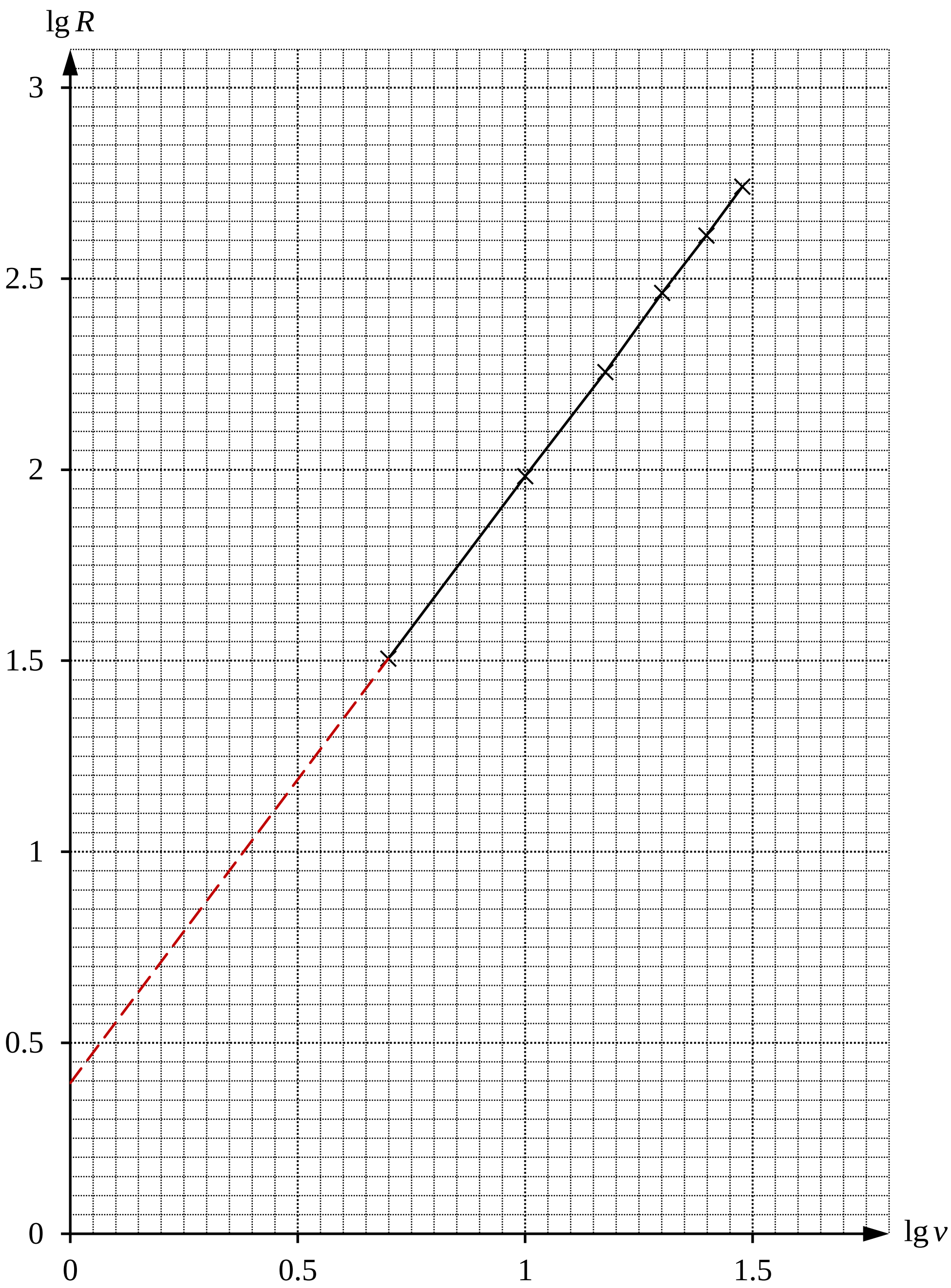
M1- Identifying vertical intercept from the graph drawn

$k = 10^{0.4}$

$= 2.51188$

$= 2.51$ (3s.f)

A1- Value of k



(ii) Use your graph to estimate the resistance for which the speed is 12 m/s. [2]

When v = 12,

$\lg v = \lg 12 = 1.07918$ M1- Finding lg 12

$\lg R = 2.1$

$R = 10^{2.1}$

$= 125.8925$

$= 126 \quad (3 \text{ s.f})$ A1- Read from graph

9 A car moves on a straight road, from a point *O*, so that *t* seconds after leaving *O*, its velocity *v* m/s is given by $v = 20\left(e^{0.2t} + 4\right)$.

(a) Find the initial acceleration of the car. [3]

$v = 20\left(e^{0.2t} + 4\right)$

$v = 20e^{0.2t} + 80$

$a = \dfrac{dv}{dt}$ M1- Attempt to differentiate velocity

$= 4e^{0.2t}$ A1

When $t = 0$,

$a = 4 \text{ m/s}^2$ A1

(b) Find the displacement of the car from *O* when $t = 5$. [4]

$v = 20e^{0.2t} + 80$

$s = \int\left(20e^{0.2t} + 80\right)dt$ M1- Attempt to integrate *v*

$= \dfrac{20e^{0.2t}}{0.2} + 80t + c$

$= 100e^{0.2t} + 80t + c$ A1- Finding *s* (exclude +*c*)

When $t = 0, s = 0$

$c = -100$ M1- Finding value of constant

$\therefore s = 100e^{0.2t} + 80t - 100$

When $t = 5$,

$s = 100e^{0.2(5)} + 80(5) - 100$

$= 571.8281 \text{ m}$

$=572$ m (3s.f) A1

(c) Will the car return to *O*? Explain and show your working clearly. [2]

$v = 20\left(e^{0.2t} + 4\right)$

For $t > 0$, $e^{0.2t} > 1$, $e^{0.2t} + 4 > 5$ and $20\left(e^{0.2t} + 4\right) > 100 > 0$.

M1- Using inequality to show that $v > 0$

Since $v > 0$ for all $t > 0$, the car will always be moving away from *O*.
Hence, the car will never return to *O*.

A1

Alternative method:
When $v = 0$,

$20\left(e^{0.2t} + 4\right) = 0$

$20e^{0.2t} = -80$

$e^{0.2t} = -4$ (No solution)

Since there is no solution when *v*=0, the car has no instantaneous rest and there is **no change in direction**. Hence the car will never return to *O*.

10 **(a)** A circle with equation $x^2 + y^2 - 2x + Ay = B$ touches the x-axis at 1 point. Find the value of B. [4]

$x^2 + y^2 - 2x + Ay = B$

$\left(x-1\right)^2 - 1 + \left(y + \frac{A}{2}\right)^2 - \frac{A^2}{4} = B$ M1-Completing the square

$\left(x-1\right)^2 + \left(y + \frac{A}{2}\right)^2 = 1 + \frac{A^2}{4} + B$

Centre $= \left(1, -\frac{A}{2}\right)$ B1- Correct x-coordinate of centre

radius $= \sqrt{\left(-1\right)^2 + \frac{A^2}{4} + B}$ M1- Finding radius

$-\frac{A}{2} = \sqrt{1 + \frac{A^2}{4} + B}$

$\frac{A^2}{4} = 1 + \frac{A^2}{4} + B$ A1

$B = -1$

Alternative method:

When $y = 0$,

$x^2 + y^2 - 2x + Ay = B$

$x^2 - 2x - B = 0$

Discriminant $= 0$

$(-2)^2 - 4(1)(-B) = 0$

$4 + 4B = 0$

$B = -1$

(b) Find the equation of the smallest possible circle which passes through the point $P(4,7)$ and has its centre C lying on the line $4y+3x-15=0$. [6]

$4y+3x-15=0$

$y=-\frac{3}{4}x+\frac{15}{4}---(1)$

Gradient $=-\frac{3}{4}$

Gradient of $PC=\frac{4}{3}$ B1- Gradient of *PC*

Equation of *PC*:

$y-7=\frac{4}{3}(x-4)$ M1- Equation of *PC*

$y=\frac{4}{3}x+\frac{5}{3}---(2)$

Sub. (2) into (1):

$-\frac{3}{4}x+\frac{15}{4}=\frac{4}{3}x+\frac{5}{3}$ M1- Elimination or substitution method

$-\frac{25}{12}x=-\frac{25}{12}$

$x=1$

Sub. $x=1$ into (1):

$y=-\frac{3}{4}(1)+\frac{15}{4}$

$y=3$

$\therefore C(1,3)$ A1- Coordinates of centre

radius $=$ length of *PC*

$=\sqrt{(4-1)^2+(7-3)^2}$ M1- Finding radius

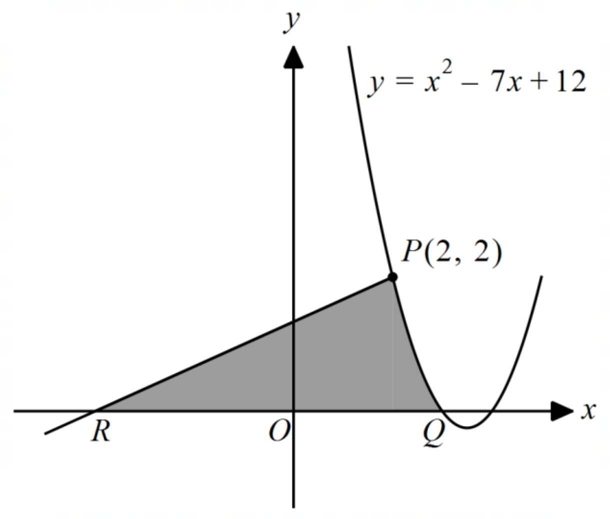
$=\sqrt{9+16}$

$=5$

Equation of circle:

$(x-1)^2+(y-3)^2=25$ A1

11 The diagram shows part of the curve $y = x^2 - 7x + 12$, cutting the x-axis at Q.
The normal to the curve at $P(2,2)$ meets the x-axis at R.



Show that the area of the shaded region bounded by the x-axis, the line PR and the curve is $6\frac{5}{6}$ units^2. [10]

$y = x^2 - 7x + 12$

$\frac{dy}{dx} = 2x - 7$ M1- Finding gradient

When $x = 2$,

$\frac{dy}{dx} = -3$

Gradient of normal $= \frac{1}{3}$ M1- Finding gradient

Equation of normal:

$$y - 2 = \frac{1}{3}(x - 2)$$

$$y = \frac{1}{3}x + \frac{4}{3}$$ M1- Equation of normal

When $y = 0$,

$\frac{1}{3}x + \frac{4}{3} = 0$

$x = -4$ A1

$\therefore R(-4,0)$

When $y = 0$,

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$ M1- Solving for x

$x = 3$ or $x = 4$ A1

$\therefore Q(3,0)$

Continuation of working space for question 11.

Area of triangle $= \frac{1}{2} \times 6 \times 2$ $= 6 \text{ units}^2$	M1- Finding area of triangle
$\int_2^3 \left(x^2 - 7x + 12\right)\, dx$ $= \left[\frac{x^3}{3} - \frac{7x^2}{2} + 12x\right]_2^3$	M1- Integrating
$= \left(\frac{27}{3} - \frac{63}{2} + 36\right) - \left(\frac{8}{3} - \frac{28}{2} + 24\right)$ $= \frac{27}{2} - \frac{38}{3}$	M1- Substitute upper and lower bound
$= \frac{5}{6} \text{ units}^2$	
Total area $= 6 + \frac{5}{6}$ $= 6\frac{5}{6} \text{units}^2$	A1- Addition to correct find total area

End of paper