STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

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EXAMINATION CODE

8	8	2	3	-	7	1	0	1

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2023

MATHEMATICS: ANALYSIS AND APPROACHES **14 August 2023**

HIGHER LEVEL **2 hours**

PAPER 1 **0900 – 1100 hrs**

Monday

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and your teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[110 marks]**.
- This question paper consists of **14** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (55 marks)

Answer **all** questions in the spaces provided.

1 **[Maximum mark: 6]**

Given that $\log_6(m+4) = -1 + \log_{36}(9) + \log_6(n)$, find the value of $2m - n$.

TURN OVER

2 **[Maximum mark: 6]**

Consider the sequence

$$\{a_n\} = \{1, -2, -3, -2, 1, 6, 13, \ldots\} \text{ for } n \in \mathbb{Z}^+ .$$

The **differences** between the successive terms, $\{d_n\}$ where $d_n = a_{n+1} - a_n$, form an arithmetic sequence.

(a) Find an expression for a_{n+1} in terms of n. [5]

(b) Hence find the value of a_{101}. [1]

TURN OVER

3 [Maximum mark: 5]

Use mathematical induction to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}$$

for all positive integers n.

TURN OVER

4 [Maximum mark: 7]

(a) Solve the inequality $\dfrac{2}{x^2-2x-8}<0$. [3]

(b) Hence or otherwise, solve the inequality $\dfrac{2}{(x-5)^2-2|x-5|-8}<0$. [4]

TURN OVER

5 **[Maximum mark: 7]**

Alison lives in Singapore and has relatives in Kuala Lumpur and Penang. Each time that she visits her relatives, she either drives or takes the plane. On each trip, Alison only visits her relatives in one city but not both.

When travelling to Kuala Lumpur, the probability that she drives is $\frac{1}{3}$, and when travelling to Penang, the probability that she takes the plane is $\frac{4}{5}$.

The probability that Alison drives to visit her relatives is $\frac{5}{18}$.

By considering a suitable probability tree diagram, find the probability that for a particular trip,

(a) Alison is visiting her relatives in Penang, [4]

(b) Alison is visiting her relatives in Penang, given that she is taking the plane. [3]

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6 [Maximum mark: 7]

Solve the equation $z^6 - (1+\mathrm{i})^6 = 0$ for $z \in \mathbb{C}$ and where $\mathrm{i}^2 = -1$.

List your answer(s), leaving them in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leq \pi$.

TURN OVER

7 **[Maximum mark: 8]**

Consider the function rule $g(x) = \mathrm{e}^{\arcsin(3x)}$.

(a) Find the maximal domain and the range of the function *g*. [4]

(b) Let $h(x) = \mathrm{e}^{\arcsin(-3x+1)}$.

(i) Describe a series of ordered transformations that map the graph of $y = g(x)$ to the graph of $y = h(x)$.

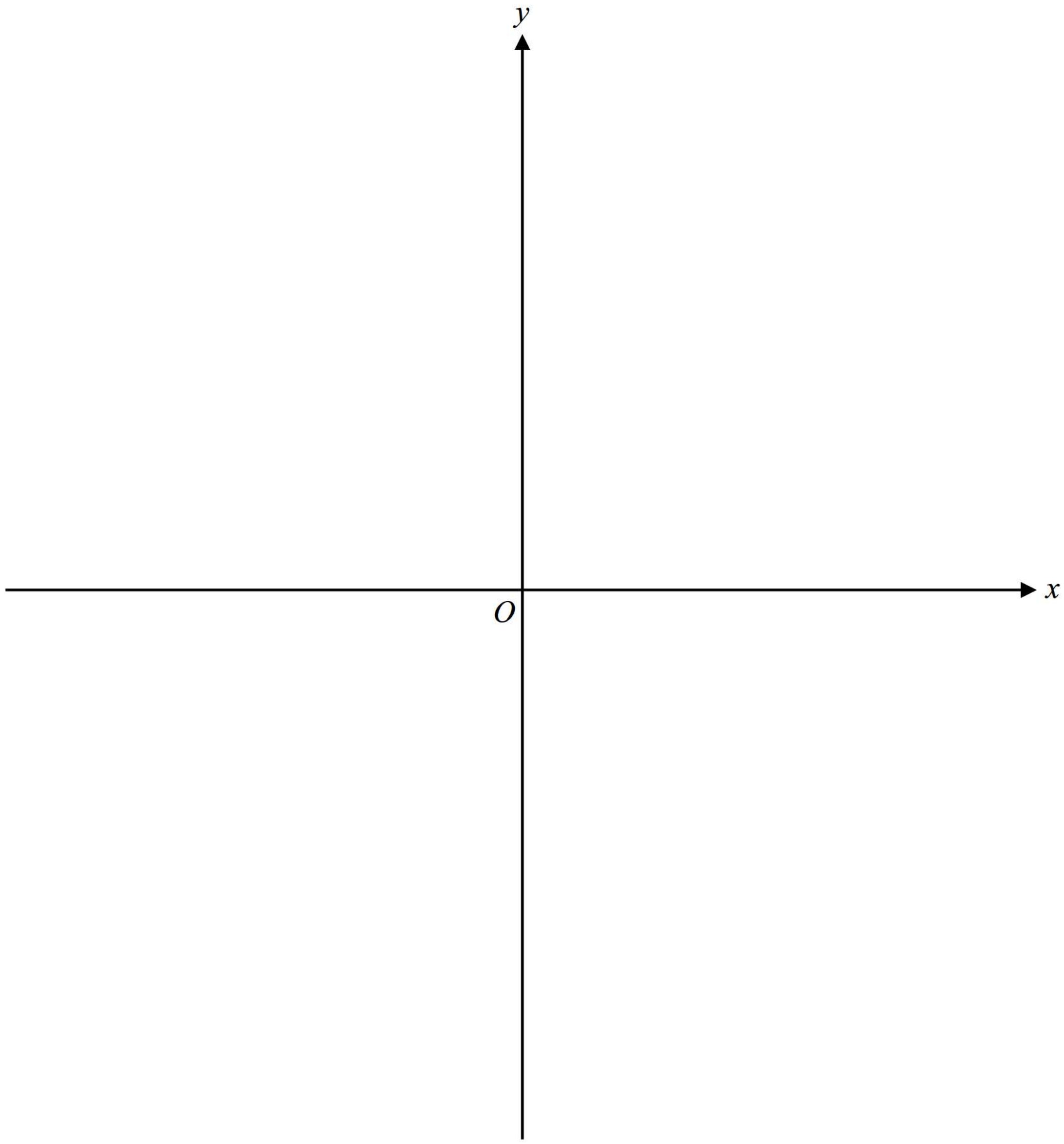
(ii) Hence find the domain of the function *h*. [4]

TURN OVER

8 **[Maximum mark: 9]**

(a) Sketch the graph of the function $f\left(x\right)=\dfrac{10}{x^2-4}$ on the axes provided, labelling clearly the coordinates of the turning point(s) and the equations of the asymptote(s), if any. [4]

(b) Find the area bounded by the graph $y=f\left(x\right)$, the x-axis and the lines $x=-1$ and $x=1$. [5]



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*Do **NOT** write solutions on this page.*

SECTION B (55 marks)

*Answer **all** questions on the writing paper provided. **Please start each question on a new page.***

9 [Maximum mark: 19]

Consider the lines

$$L_1 : \frac{x-2}{2} = y - 2 = \frac{z-1}{2}$$

and

$$L_2 : \frac{x+1}{a} = \frac{y+3}{2} = \frac{z}{b}$$

where a, b and c are constants.

(a) Find the value of a and of b such that L_1 and L_2 are parallel. [3]

(b) Show that L_1 does not intersect L_2 when $a = b$. [3]

Suppose that $a = -b$ and that L_1 and L_2 intersect.

(c) Show that $a = \frac{1}{2}$. [4]

Suppose that $a = \frac{1}{2}$ and $b = -\frac{1}{2}$. Let Π_1 be the plane that contains both L_1 and L_2.

(d) Verify that the vector $-9\boldsymbol{i} + 4\boldsymbol{j} + 7\boldsymbol{k}$ is normal to Π_1. [3]

(e) Find the Cartesian equation of Π_1. [3]

Consider another plane Π_2, with Cartesian equation $-9x + 4y + 7z = 2$.

(f) Find the shortest distance between Π_2 and L_2. [3]

TURN OVER

*Do **NOT** write solutions on this page.*

10 [Maximum mark: 20]

Let $f(x) = (1 - x^2)^{-n}$, for some $n \in \mathbb{N}$.

(a) Find $f'(x)$. Leave your answer in terms of n. [2]

(b) Show that $f''(x) = \dfrac{2n(1 + x^2 + 2nx^2)}{(1 - x^2)^{n+2}}$. [3]

(c) Hence, find the Maclaurin expansion of $f(x)$, in ascending powers of x, as far as the term in x^2. [3]

(d) Find the binomial expansion of $(1 - x^2)^{-1}$, in ascending powers of x, as far as the term in x^4. [3]

Consider $(1 - x^2)^{-1} = \dfrac{1}{1 - x^2}$ as the sum of an infinite geometric series.

(e) Write down the first four terms of this infinite geometric series. [2]

(f) Find the values of x for which this infinite sum is valid. [2]

(g) By comparing your result in (e) with the Maclaurin expansion of $f(x)$, find the value of $f^{(4)}(0)$. [3]

(h) Deduce the value of $f^{(m)}(0)$ when m is even and the value of $f^{(m)}(0)$ when m is odd. [2]

TURN OVER

*Do **NOT** write solutions on this page.*

11 [Maximum mark: 16]

Let $g(x) = \dfrac{x\cos x + \pi}{x - \pi}, -\pi < x < \pi.$

(a) Find the y-intercept of $y = g(x)$. [1]

(b) By considering the range of $\cos x$, show that $y = g(x)$ has no x-intercept. [3]

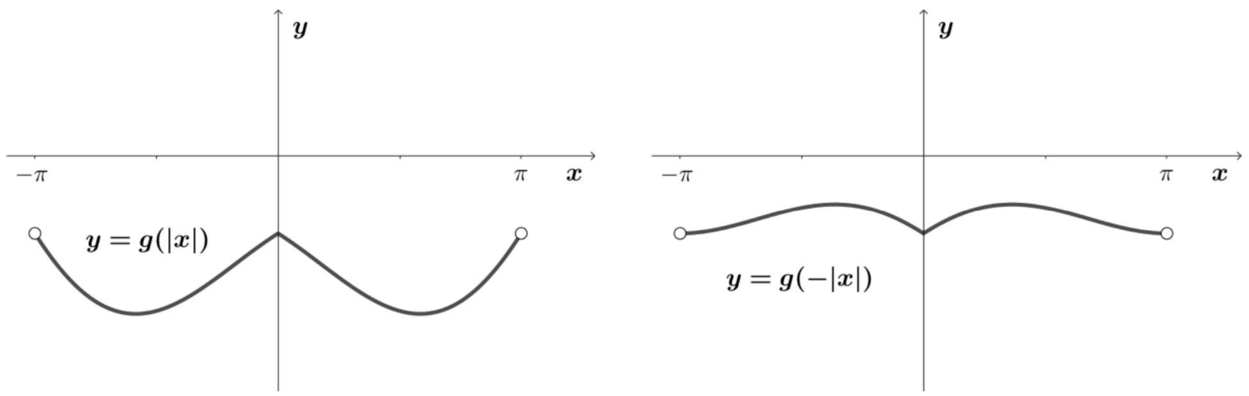
(c) Find the value of $\lim\limits_{x \to -\pi} g(x)$. [2]

(d) **(i)** Explain why $\lim\limits_{x \to \pi} g(x)$ can be found by using l'Hôpital's Rule.

(ii) Hence find $\lim\limits_{x \to \pi} g(x)$. [4]

(e) By using the answer in (d), explain why there is no vertical asymptote at $x = \pi$. [1]

The graphs of $y = g(|x|)$ and $y = g(-|x|)$ are given below.



(f) Sketch the graph of $y = g(x)$, clearly indicating the coordinates of the y-intercept and of the endpoints. [5]

End of Paper