H3 Mathematics Problem Solving

Lecture 2: Accessing Mathematical Resources

Dr. Ho Weng Kin

Reconnecting to Last Lesson

Last week:

- What is a genuine mathematical problem?
- Pólya's four stages:
  1. Understand
  2. Devise a plan
  3. Carry out
  4. Look back

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- What is a genuine mathematical problem?
- Pólya's four stages:
    1. Understand
    2. Devise a plan
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    4. Look back

Today's focus:

How do we actually solve problems?

A Key Idea for Today

Core Message

Problem solving does not happen in a vacuum.

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- We rely on **mathematical resources**
- We use **heuristics** to activate them
- We must learn to **recognise structure**

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Key Insight

Do not wait for a problem to match a theorem.
Force the problem to match the theorem.

Book link: Chapter 2: Accessing mathematical resources; Chapter 3: Deploying problem solving heuristics

Homework (1) Discussion

Problem

Let a, b and c be positive real numbers.
By using the results in parts (a)

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$$

and (b)

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$

prove that

$$\frac{9}{a+b+c}\leq 2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right)\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Book link: Chapter 1, Exercises; Chapter 2, “Accessing mathematical resources”

Homework (1) Discussion

If we take each of these sums

$$\frac{1}{a}+\frac{1}{b},\ \frac{1}{b}+\frac{1}{c},\ \frac{1}{a}+\frac{1}{c}$$

one at a time and apply (a), we get

- $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$;
- $\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}$;
- $\frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c}$.

Homework (1) Discussion

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- $\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}$;
- $\frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c}$.

Summing up all three lines, we have $2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 4\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}\right)$.

Homework (1) Discussion

To prove that

$$2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right)\geq\frac{9}{a+b+c},$$

you stare hard at part (b), which is:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9.$$

Homework (1) Discussion

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What do you substitute x, y and z for?

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you stare hard at part (b), which is:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

Let $x=a+b$, $y=a+c$ and $z=b+c$.

Homework (1) Discussion

Thus,

$$[(a+b)+(a+c)+(b+c)]\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \geq 9.$$

That leads us to

$$2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \geq \frac{9}{a+b+c},$$

which is what we wanted to prove.

Homework (1) Discussion

Let us move to the next question.

Homework (1) Discussion

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For $a, b, c > 0$ with $abc = 1$, prove:

$$(1+a)^2(1+b)^3(1+c)^4 > 256$$

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Observation:

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Homework (1) Discussion

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For $a, b, c > 0$ with $abc = 1$, prove:

$$(1+a)^2(1+b)^3(1+c)^4 > 256$$

Observation:

$$(1+a)^2, \quad (1+b)^3, \quad (1+c)^4$$

Heuristic

Look for patterns and structure.

Activating a Resource

Recall:

$$((n-1)x+y)^n \geq n^n x^{n-1} y$$

Book link: Chapter 2, Section 2: Accessing mathematical resources

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- Choose n appropriately

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Heuristic: Template Matching

We rewrite the problem to fit a known inequality.

Applying the Idea

Case 1: $n = 2, x = 1, y = a$

$$(1 + a)^2 \geq 4a$$

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Case 3: $n = 4, x = \frac{1}{3}, y = c$

$$(1+c)^4 \geq \frac{256}{27}c$$

Multiply:

$$(1+a)^2(1+b)^3(1+c)^4 \geq 256abc = 256$$

Book link: Chapter 1, Example on inequality; Chapter 2, Section 2

Stage 4: Look Back

We obtained:

$$\geq 256$$

But the question asks:

$$> 256$$

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But then:

$$abc = \frac{1}{6} \neq 1$$

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Conclusion

Equality is impossible $\Rightarrow$ strict inequality holds.

Book link: Chapter 1, Section 1.4: Looking back

Meta Reflection

What did we actually do?

Meta Reflection

What did we actually do?

- Recognised structure
- Forced expressions into a known inequality
- Applied AM-GM strategically
- Checked equality conditions

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Lesson

Looking back is not optional — it completes the solution.

When We Get Stuck

Sometimes, our current tools are not enough.

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Just-in-Time Teaching

Introduce a new tool only when it is needed.

Cauchy-Schwarz inequality

Augustin-Louis Cauchy came up with our present version involving real numbers, while Hermann Schwarz was the first to give a formal proof for the integral version.

Figure: Augustin-Louis Cauchy & Hermann Schwarz

Cauchy-Schwarz Inequality

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ it holds that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

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Interpretation: $n = 3$

- Dot product $\leq$ product of lengths

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- Dot product $\leq$ product of lengths

When to use it

- Sums of products
- Symmetric expressions

Book link: Chapter 1, Section 2; Chapter 2, Section 4

Application

Prove:

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Idea:

- Let $b_i = 1$
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Application

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Idea:

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Key Move

Turn a sum into a dot product.

$$a_1 \cdot 1 + \cdots + a_n \cdot 1 \leq \sqrt{(\underbrace{1^2 + \cdots + 1^2}_{n \text{ copies}})(a_1^2 + \cdots + a_n^2)}.$$

Application

Can we go a bit further to apply the above inequality

$$a_1 + \cdots + a_n \leq \sqrt{n(a_1^2 + \cdots + a_n^2)}$$

to justify that for any positive real numbers x, y and z,

$$\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} + \sqrt{\frac{z+x}{x+y+z}} \leq \sqrt{6}?$$

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What are your first thoughts?

Who is he?

Figure: Picture of Terence Tao (aged 10) with Paul Erdös

Homework (1) Discussion

Let us talk about Terence Tao's problem he posed on his blog:

Problem statement

I was recently at an international airport, trying to get from one end of a very long terminal to another. It inspired in me the following simple mathematics puzzle, which I thought I would share here: Suppose you are trying to get from one end A of a terminal to the other end B. (For simplicity, assume the terminal is a one-dimensional line segment.) Some portions of the terminal have moving walkways, other portions do not. Your walking speed is a constant, but while on a walkway, it is boosted by the speed of the walkway. Your objective is to get from A to B in the shortest time possible. Suppose you need to pause for some period of time, say to tie your shoe. Is it more efficient to do so while on a walkway, or off the walkway? Assume the period of time required to pause is the same in both cases.

Terence Tao's Airport Problem: Solution

Answer

It is more efficient to pause while on the moving walkway.

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- walking speed is v;
- walkway speed is $u > 0$;
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If you pause off the walkway, distance gained during the pause is

$$0.$$

If you pause on the walkway, distance gained during the pause is

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Key Insight

Even when you stop walking, the walkway still moves you forward.

Reflection

So far, the key ideas we have come across are:

Reflection

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- Problem solving requires resources
- Heuristics help us activate them
- Structure is everything
- Looking back reveals deeper truth

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- Problem solving requires resources
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We are learning how to **think**, not just solve.

What Did We Just Do?

Let us pause.

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In the previous problems, we:

- Substituted cleverly
- Matched expressions to known results
- Looked for structure

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Key Question

Are these random tricks. . . or something systematic?

What Did We Just Do?

Let us pause.

In the previous problems, we:

- Substituted cleverly
- Matched expressions to known results
- Looked for structure

Key Question

Are these random tricks... or something systematic?

They are called heuristics.

Book link: Chapter 3: Deploying problem solving heuristics

What is a Heuristic?

Definition

A heuristic is a **problem solving strategy** or **rule of thumb** that guides us toward a solution.

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- Not guaranteed to work
- But increases chances of success
- Comes from experience and insight

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Heuristics is a Greek word that means 'serving to discover'. Think Eureka!

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- But increases chances of success
- Comes from experience and insight

Key Idea

Good problem solvers are not smarter.
They have **more heuristics** and know **when to use them**.

A Toolbox of Heuristics

1. Restate the problem
2. Work backwards
3. Aim for sub-goals
4. Divide into cases
5. Consider a simpler problem
6. Consider a more general case
7. Act it out
8. Guess and check
9. Make a systematic list
10. Look for patterns
11. Use algebra
12. Use suitable representation
13. Use suitable notation

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Important

Do NOT memorise this list mechanically.
Learn to **recognise when to use them**.

Heuristics in Action

Reality

We rarely use just one heuristic.

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We combine them.

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- Simplify + pattern recognition
- Representation + case analysis
- Backward thinking + algebra

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Today

We study ONE problem deeply.

The Princess Problem

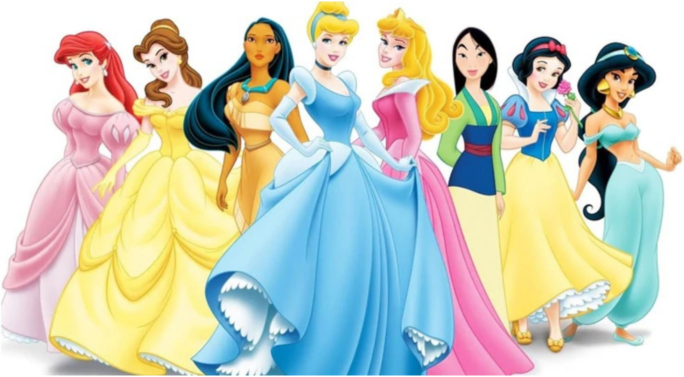

Figure: The Princess Problem

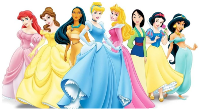

Figure: The Princess Problem

Problem
There are 17 rooms in a row.
Each day:

- The princess moves to an adjacent room
- The prince checks one room

The Princess Problem

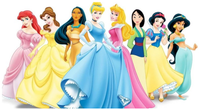

Figure: The Princess Problem

Problem

There are 17 rooms in a row.

Each day:

- The princess moves to an adjacent room
- The prince checks one room

Can the prince guarantee to meet the princess within 30 days?

Book link: Chapter 3, Section 2.1: The Princess Problem

Heuristic: Simplify the Problem

Idea

17 rooms is too complicated.

Heuristic: Simplify the Problem

Idea

17 rooms is too complicated.

Start with 3 rooms.

Heuristic: Simplify the Problem

Idea

17 rooms is too complicated.

Start with 3 rooms.

Heuristic

Consider a simpler problem

Heuristic: Use Representation

Rooms: 1 2 3

Heuristic: Use Representation

Rooms: 1 2 3

- Princess moves each day
- Prince chooses a room

Heuristic: Use Representation

Rooms: 1 2 3

- Princess moves each day
- Prince chooses a room

Heuristic

Draw a diagram / model the situation

Heuristic: Look for Patterns

Naive strategy fails.

Heuristic: Look for Patterns

Naive strategy fails.

Observation

Prince and princess are always on opposite parity:

- Odd vs Even rooms

Heuristic: Look for Patterns

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Observation

Prince and princess are always on opposite parity:

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Deep Insight: Big Idea of Invariance

This is an **invariant**.

Heuristic: Divide into Cases

- Case 1: Princess starts in odd room
- Case 2: Princess starts in even room

Heuristic: Divide into Cases

- Case 1: Princess starts in odd room
- Case 2: Princess starts in even room

Strategy

Visit Room 2, then stay.

Heuristic: Divide into Cases

- Case 1: Princess starts in odd room
- Case 2: Princess starts in even room

Strategy

Visit Room 2, then stay.

Result

Guaranteed capture.

Generalising the Strategy

- Sweep from Room 2 to Room 16
- Pause to switch parity
- Sweep back

Generalising the Strategy

- Sweep from Room 2 to Room 16
- Pause to switch parity
- Sweep back

Heuristic

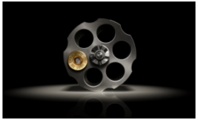
Generalise from small case

Russian Roulette Problem

MA | Mathematically Appropriate Content

Depiction of firearms and implied shooting

Russian Roulette Problem

Click on the barrel! 

Russian Roulette Problem

Two bullets in consecutive chambers.

Russian Roulette Problem

Two bullets in consecutive chambers.
First shot: SAFE.

Russian Roulette Problem

Two bullets in consecutive chambers.
First shot: SAFE.
Second player:

- Spin again?
- Or pull immediately?

Book link: Chapter 3, Section 2.2: The Russian Roulette Problem

Heuristic: Divide into Cases

Case 1: Spin

$$P(\text{survival}) = \frac{4}{6} = \frac{2}{3}$$

Heuristic: Divide into Cases

Case 1: Spin

$$P(\text{survival}) = \frac{4}{6} = \frac{2}{3}$$

Case 2: Do not spin

$$P(\text{survival}) = \frac{3}{4}$$

Heuristic: Divide into Cases

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$$P(\textsf{survival}) = \frac{4}{6} = \frac{2}{3}$$

Case 2: Do not spin

$$P(\textsf{survival}) = \frac{3}{4}$$

- We know the first shot was safe $\Rightarrow$ current chamber is empty
- There are 4 empty chambers in total
- Only **one** of these is followed by a bullet (since bullets are consecutive)

Heuristic: Divide into Cases

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Key Insight

Among the 4 possible empty chambers, only 1 leads to danger next.
So survival probability is $\frac{3}{4}$.

Heuristic: Divide into Cases

Case 1: Spin

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Conclusion

Do NOT spin.

Key Insight

What changed?

Information from first event.

Key Insight

What changed?

Information from first event.

Heuristic

Use **conditional reasoning**

Key Insight

What changed?

Information from first event.

Heuristic

Use **conditional reasoning**

Information changes probability.

Check and Expand

Problem extension

If one survives the second pull of the trigger and the revolver is passed to the first player, what should be the most strategic choice? Just pull or Spin & Pull?

From Chapter 3: Deploying problem solving heuristics

1. H3 2018 Question 4: T-shirt problem
2. H3 2017 Question 5: Distributing objects into identical boxes

Homework (2): Chapter 3

From Chapter 3: Deploying problem solving heuristics

1. H3 2018 Question 4: T-shirt problem
2. H3 2017 Question 5: Distributing objects into identical boxes

Instruction

For each problem, write down both the solution and the heuristics used.

Book link: Chapter 3, Exercises

Summary: What Have We Learned?

- Problem solving does not happen in isolation
- We must **activate mathematical resources**
- Heuristics guide us when no method is obvious
- Structure determines strategy

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- Structure determines strategy

Big Idea

Do not wait for the right method.
Shape the problem until a method appears.

Book link: Chapter 2; Chapter 3

Summary: Developing as a Problem Solver

- Recognise patterns and structure
- Build a repertoire of heuristics
- Learn to connect problems to known ideas
- Reflect and refine through looking back

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- Recognise patterns and structure
- Build a repertoire of heuristics
- Learn to connect problems to known ideas
- Reflect and refine through looking back

We are not just solving problems.
We are learning how to think.

Book link: Chapter 3; Chapter 4 (Look back, check and expand)

References

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