Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2017

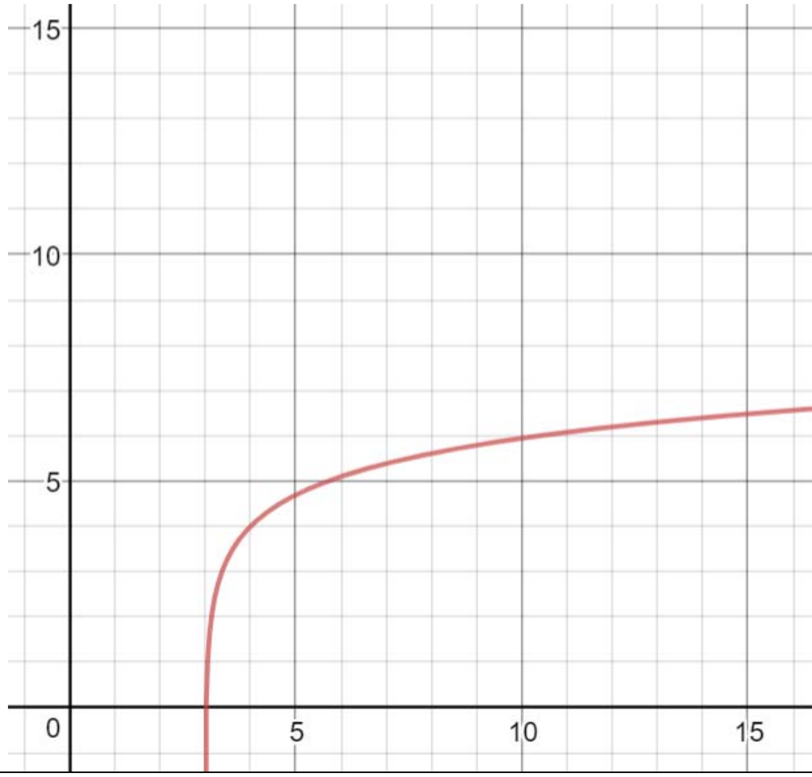
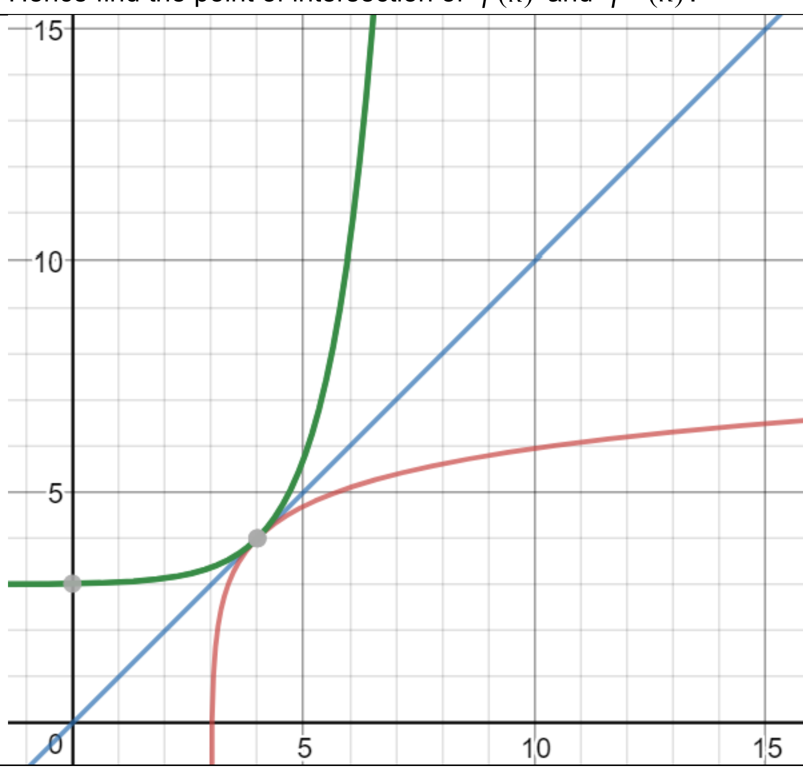
YEAR 5 IB DIPLOMA PROGRAMME

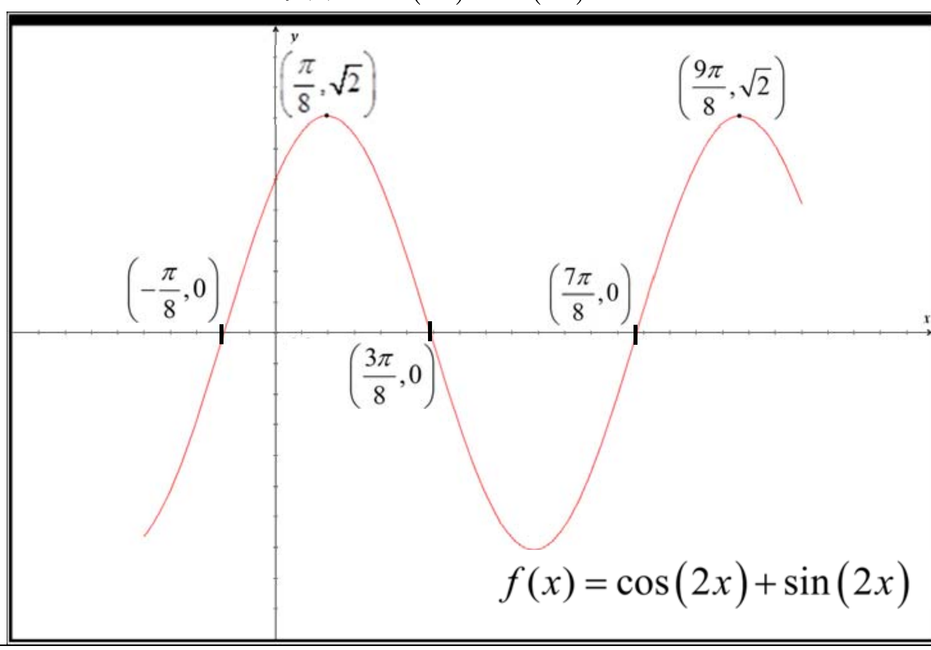
MATHEMATICS

HIGHER LEVEL

Paper 1

Qn	Solution
1. (a)	Find the term independent of x in the expansion of $\left(x^2+\dfrac{2}{x}\right)^6$.
(b)	Prove that $\begin{pmatrix} n \\ r \end{pmatrix} \times \dfrac{n-r}{r+1} = \begin{pmatrix} n \\ r+1 \end{pmatrix}$.
	$T_{r+1} = {}^6C_r \times x^{2(6-r)} \times \left(\dfrac{2}{x}\right)^r = {}^6C_r \times 2^r \times x^{12-3r}$ $12-3r=0 \quad \Rightarrow r=4$ $\text{Independent term} = {}^6C_4 \times 2^4 = \dfrac{6\times5\times4\times3}{4\times3\times2\times1} \times 2^4 = 30\times8 = 240$
	$LHS = \begin{pmatrix} n \\ r \end{pmatrix} \dfrac{n-r}{r+1} = \dfrac{n!}{r!(n-r)!} \times \dfrac{n-r}{r+1}$ $= \dfrac{n!}{(r+1)!(n-(r+1))!}$ $= \begin{pmatrix} n \\ r+1 \end{pmatrix} = RHS$

Qn	Solution
2.	The diagram shows the graph of f . 
(a)	On the same diagram, sketch the graph of f^{-1} .
(b)	Hence find the point of intersection of $f(x)$ and $f^{-1}(x)$.
	

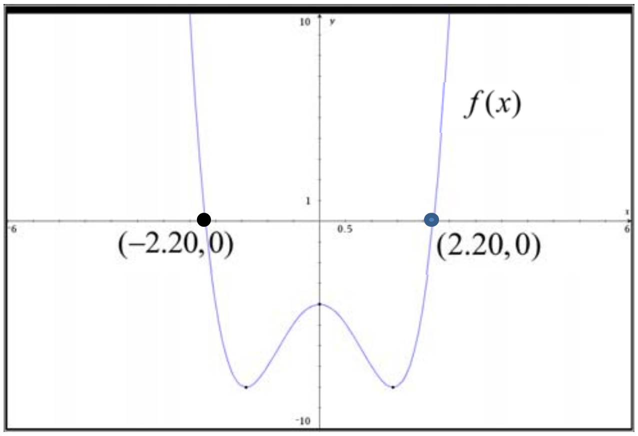
Qn	Solution
	At the intersection of $f(x)$ and $f^{-1}(x)$, $f(x)=f^{-1}(x)=x$. Point of intersection is (4, 4).
3.	The cubic polynomial $f(x)$ and quadratic polynomial $p(x)$ are related by the following equation: $f(x)=p(x)\cdot(x-1)+4$. The roots of $f(x)$ are at $x=2$ and $x=3$. Find $f(x)=x^3+Ax^2+Bx+C$, where A, B and C are constants to be determined.
	$f(x)=(x-1)\cdot p(x)+4=(x-2)(x-3)(x-k)$ $f(1)=4=2(1-k) \quad \Rightarrow k=-1$ $f(x)=x^3+Ax^2+Bx+C=(x-2)(x-3)(x+1)$ $A=-(2+3-1)=-4$ $C=-(2\times3\times(-1))=6$ $B=(-2)(-3)+(-3)(1)+(1)(-2)=1$ $\therefore f(x)=x^3-4x^2+x+6$
4.	The graph of function $f(x)=\cos(2x)+\sin(2x)$ for $-1\le x\le 4$ is shown below. 
(a)	The function f can also be expressed as $f(x)=a\sin(2(x+b))$ where $a\in\mathbb{R}$ and $0\le b\le 2$. Find the value of a and b.
(b)	Find $f'(x)$, hence show that $\cos(2x)-\sin(2x)=\sqrt{2}\cos\left(2\left(x+\frac{\pi}{8}\right)\right)$.
	$f(x)=a\sin(2(x+b))$ Amplitude of f(x) = $\sqrt{2}$ = a Function is translated b units, b = $\frac{\pi}{8}$

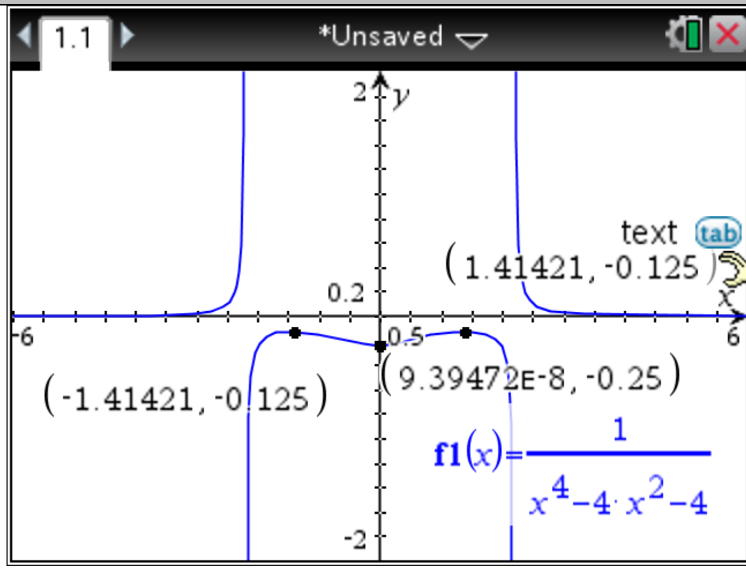
Qn	Solution
	Differentiating f(x), $f'(x) = -2\sin(2x) + 2\cos(2x)$ $f'(x) = \sqrt{2}\cos\left(2\left(x+\frac{\pi}{8}\right)\right)\times 2$ $\cos(2x) - \sin(2x) = \sqrt{2}\cos\left(2\left(x+\frac{\pi}{8}\right)\right)$
5.	Determine the relationship between the following lines. Explain your answer clearly. $l_1 : \mathbf{r} = \begin{pmatrix} 11 \\ 8 \\ 3 \end{pmatrix} + k\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, k \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} + h\begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, h \in \mathbb{R}.$
	Their direction vectors are not parallel, they are not parallel lines. $\begin{pmatrix} 11+2k \\ 8+5k \\ 3+4k \end{pmatrix} = \begin{pmatrix} -2+3h \\ 1-h \\ 9+5h \end{pmatrix} \Rightarrow (1)\,\&\,(3) \quad 19 = -13 + h$ $\Rightarrow h = 32, k = \frac{83}{2}$ $(2): LHS = 8 + 5\times\frac{83}{2}$ $RHS = 1 - 32$ The lines do not intersect. The lines are skew.
6.	Use mathematical induction to prove that $7-4^n$ is divisible by 3 for $n \in \mathbb{Z}^+$.

Qn	Solution
	Let Pn be the proposition: $7-4^n=3p, \quad p\in\mathbb{Z}$ To prove P1 is true, $7-4=3$. P1 is true. Assume Pn is true for n=k: $7-4^k=3q, \quad q\in\mathbb{Z}$ To prove Pk+1 is true: $7-4^{k+1}=3r, \quad r\in\mathbb{Z}$ $LHS=7-4^{k+1}=7-4(4^k)=7-4\left(7-3q\right)$ $=7-4\times 7+4\times 3q$ $=3\left(4q-7\right)=3r$ Since P1 is true and Pk is true $\Rightarrow$ Pk+1 is true, $7-4^n$ is divisible by 3 for $n\in\mathbb{Z}^+$.
7.	The functions f and g are defined by $f\left(x\right)=x^2+1$, $x\in\mathbb{R}$ and $g\left(x\right)=-\sqrt{x-1}$, $x\in\mathbb{R}, x\geq 1$.
(a)	State the ranges for f and g.
(b)	Give a reason why fg exists.
(c)	Find the composite function fg and state its domain.
	$R_f=[1,\infty[$ $R_g=]-\infty,0]$
	$R_g=]-\infty,0]\subset D_f=]-\infty,\infty[$ Hence fg exists.
	$f\left(g(x)\right)=\left(-\sqrt{x-1}\right)^2+1=\pm\left(x-1\right)+1$ Since $x\in\mathbb{R}, x\geq 1$ $=\left(x-1\right)+1=x$ $D_{fg}=D_g=[1,\infty[$
8.	**(ai)** Given the equation of the curve $x^2-2xy+2y^2=4$, find $\frac{dy}{dx}$ in terms of x and y. **(aii)** Hence find the value(s) of x when the tangent is parallel to the x-axis.
	(b) Differentiate $\arctan\sqrt{x}$ with respect to x.

Qn	Solution
	$x^2-2xy+2y^2=4$ ai) $2x-2x\dfrac{dy}{dx}-2y+4y\dfrac{dy}{dx}=0$ $\dfrac{dy}{dx}=\dfrac{y-x}{2y-x}$ aii) When the tangent if parallel to the x-axis, $\dfrac{dy}{dx}=0$. $\dfrac{dy}{dx}=\dfrac{y-x}{2y-x}=0$ $x=y$ $x^2-2xy+2y^2=4$ $\Rightarrow x^2-2x^2+2x^2=4$ $x=\pm 2$
	b) $\dfrac{d}{dx}\tan^{-1}\sqrt{x}=\dfrac{1}{1+\left(\sqrt{x}\right)^2}\times\dfrac{1}{2\sqrt{x}}$ Application of arctanx formula and chain rule. $=\dfrac{1}{2\sqrt{x}\left(1+x\right)}$
	Section B
9.	Consider the function $f(x)=\dfrac{ax+b}{cx+d}$, $x\in\mathbb{R}$, $x\neq-\dfrac{d}{c}$ for non-zero constants a , b , c and d .
(a) i)	Find an expression for $f'\left(x\right)$.
ii)	Given that $ad>bc$, show that $f\left(x\right)$ is an increasing function for all values of $x\in\mathbb{R}$, $x\neq-\dfrac{d}{c}$.
	$f'(x)=\dfrac{a(cx+d)-c(ax+b)}{(cx+d)^2}$ $=\dfrac{ad-bc}{(cx+d)^2}$, $x\neq-\dfrac{d}{c}$
	Since $ad>bc, ad-bc>0$, $f'(x)=\dfrac{ad-bc}{(cx+d)^2}>0$, $f\left(x\right)$ is an increasing function.
(b)	Given $a=2$, $b=0$, $c=3$ and $d=-2$, find $f^{-1}\left(x\right)$ for $x\in\mathbb{R}, x\neq\dfrac{2}{3}$.
(i)	Hence or otherwise show that $f^2\left(x\right)=x$ and find $f^{201}\left(x\right)$.
(ii)	With the given values of a,b,c and d , solve the inequality $f(x)\leq x$.

Qn	Solution
	Let $y=\dfrac{2x}{3x-2}$ $2x=3xy-2y$ $x=\dfrac{2y}{3y-2}$ $f^{-1}(x)=\dfrac{2x}{3x-2}$
	$f(x)=\dfrac{2x}{3x-2}=f^{-1}(x)$ $f\left(f(x)\right)=f\left(f^{-1}(x)\right)=x$
	$f^{201}\left(x\right)=f^{2\times 100}(f(x))=f(x)=\dfrac{2x}{3x-2}$
	$\dfrac{2x}{3x-2}\le x$ $\dfrac{2x-x(3x-2)}{3x-2}\le 0$ $\dfrac{4x-3x^2}{3x-2}\le 0$ $\left(4x-3x^2\right)\left(3x-2\right)\le 0$ Roots at $x=0, x=\dfrac{2}{3}, x=\dfrac{4}{3}$. $0\le x<\dfrac{2}{3},\quad x\ge\dfrac{4}{3}$
(c)	The function h is defined by $h\left(x\right)=p\sin x+qx+r$, $x\in\mathbb{R}$, where p,q and r are real constants. Given that h is an odd function, find the value of r .
	$h\left(x\right)=-h(-x)$ $p\sin x+qx+r=-p\sin(-x)-q(-x)-r$ $p\sin x+qx+r=p\sin(x)+qx-r$ $\Rightarrow r=0$

Qn	Solution
10.	The graph of a polynomial function $f(x)=x^4-4x^2-4$ of degree four is shown below. $f(x)$ has roots at $(-2.20,0)$ and $(2.20,0)$. 
(a)	Determine the interval(s) on the domain of f where f is concave downwards.
(b)	$f(x)$ is translated by h units in the positive y-direction such that the equation $f(x)=0$ will have four real and distinct roots. Find all possible value(s) of h.
	$f(x)=x^4-4x^2-4$ $f'(x)=4x^3-8x$ $f''(x)=12x^2-8<0$ $-\sqrt{\frac{2}{3}}<x<\sqrt{\frac{2}{3}}$
	$f(x)=x^4-4x^2-4$ $f'(x)=4x^3-8x=0$ $x(x^2-2)=0$ $f(x)$ has turning points at $x=0$, $x=\pm\sqrt{2}$. When $x=0$, $y=-4$. When $x=\pm\sqrt{2}$, $y=2^2-4(2)-4=-8$. $\therefore 4<h<8$
(b)	Sketch the graph of $\frac{1}{f(x)}$ for the function $f(x)$ shown above. State clearly the coordinates of any asymptotes, intercepts and turning points on the graph.

Qn	Solution
	 Vertical asymptotes: $x=a$ and $x=-a$ Horizontal asymptote: $y=0$ Min point/ y-intercept: $(0,-0.25)$ Max points: $(-\sqrt{2},-\frac{1}{8})$ and $(\sqrt{2},-\frac{1}{8})$
11.	Let $p(x)=8x^3-8x^2-6x+10, x\in\mathbb{C}$.
(a)	The roots of the equation $p(x)=0$ are α , β and γ . Find the values of
(i)	$\alpha+\beta+\gamma$;
(ii)	$\alpha\beta\gamma$,
(iii)	$\alpha\beta+\alpha\gamma+\beta\gamma$.
	$p(x)=8x^3-8x^2-6x+10, x\in\mathbb{R}$ Sum of roots = $\frac{8}{8}=1$ Product of roots = $\frac{-10}{8}=\frac{-5}{4}$ $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{-6}{8}=-\frac{3}{4}$
(b)	Find all three roots of $p(x)=0$. Show that the roots form a triangle with an area of 1 unit^2 when plotted on an Argand diagram.

Qn	Solution
	$p(x)=8x^3-8x^2-6x+10$ $p(-1)=0$ $8x^3-8x^2-6x+10=(x+1)(Ax^2+Bx+C)$ $A=8$ $C=10$ $-6=C+B \quad \Rightarrow B=-16$ $8x^2-16x+10=0$ $4x^2-8x+5=0$ $x=\dfrac{8\pm\sqrt{64-4\times 20}}{8}=1\pm\dfrac{1}{2}i$ Vertices of triangle $(-1,0),(1,0.5),(1,-0.5)$. Area of triangle = $\dfrac{1}{2}\times 2\times 1=1$
(c)	A new polynomial is defined by $q(x)=p(x-2)$.
(i)	Find the value of the sum of roots of the equation $q(x)=0$.
(ii)	Find the value of the product of roots of the equation $q(x)=0$.
	Since $\begin{aligned}q(x)=p(x-2)&=((x-2)-\alpha)((x-2)-\beta)((x-2)-\gamma)\\&=(x-(2+\alpha))(x-(2+\beta))(x-(2+\gamma))\end{aligned}$ Sum of roots = $(\alpha+\beta+\gamma)+3(2)=7$
	Product of roots = $(2+\alpha)(2+\beta)(2+\gamma)=2^3+4(\alpha+\beta+\gamma)+2(\alpha\beta+\beta\gamma+\alpha\gamma)+\alpha\beta\gamma$ $=8+4\times1+2\left(\dfrac{-3}{4}\right)-\dfrac{5}{4}=9\dfrac{1}{4}$