Topic: Matrices

Question 1 [2008 EOY, Q18 OR]	marks
(a) If $\begin{pmatrix} 2a & 3 \\ -a & a \end{pmatrix}$ is singular, find the possible values of *a*. **Solution:** $\det\begin{pmatrix} 2a & 3 \\ -a & a \end{pmatrix} = 0 \Rightarrow 2a^2 + 3a = 0 \Rightarrow a(2a+3) = 0$ Hence $a = 0$ or $a = -\frac{3}{2}$	[3]
(b) (i) It is given that $2ax + 3y = 15$ and $ay - ax = k$, where *a* and *k* are real constants and $a \neq 0$. **Solution:** Write $2ax + 3y = 15$ and $ay - ax = k$ in matrix form: $\begin{pmatrix} 2a & 3 \\ -a & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ k \end{pmatrix}$ For infinite many solutions, $a = -\frac{3}{2}$ so $\begin{pmatrix} -3 & 3 \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ k \end{pmatrix}$ $\Rightarrow -3 : \frac{3}{2} = 15 : k \Rightarrow k = -\frac{15}{2}$	[2]
(b) (ii) Solve the simultaneous equations for *x* and *y* if $a = -1$ and $k = -5$. **Solution:** When $a = -1$ and $k = -5$, $\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ -5 \end{pmatrix}$ hence $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 15 \\ -5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 15 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ i.e. $x = 0$ and $y = 5$	[3]

Question 2 [2009 EOY, Q6]	marks
Solve the following simultaneous equations using the matrix method $5x+3y+10=7x+7y-30=8x+6y-29$. **Solution:** $2x+4y=40$ $x+2y=20 \rightarrow (1)$ $x-y=-1 \rightarrow (2)$ In Matrix Form: $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 20 \\ -1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 20 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}=\frac{1}{-3}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}\begin{pmatrix} 20 \\ -1 \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 6 \\ 7 \end{pmatrix}$ => x = 6, y = 7	[5]

Question 3 [2010 EOY, Q17 OR]	marks
(a) If $\begin{pmatrix} 3a & 2 \\ a & -a \end{pmatrix}$ is singular, find the possible values of a. **Solution:** $\det\begin{pmatrix} 3a & 2 \\ a & -a \end{pmatrix}=0 \Rightarrow -3a^2-2a=0$ $\Rightarrow a(3a+2)=0$ $\Rightarrow a=0 \text{ or } a=-\frac{2}{3}$	[3]

(b) It is given that $3ax+2y=8$ and $ax-ay=k$, where *a* and *k* are real constants and $a \neq 0$.	
(i) Determine the value of *k* if there are infinitely many solutions to the pair of simultaneous equations. **Solution:** Must be singular so $a=-\frac{2}{3}$ (since $a \neq 0$) So $2 \div \frac{2}{3} = 8 \div k \Rightarrow k=\frac{8}{3}$	[2]
(ii) Use the matrix method to solve the simultaneous equations for *x* and *y* if $a=1$ and $k=6$. **Solution:** $3x+2y=8$ and $x-y=6$ so $\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ Hence $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ $=\frac{1}{-5}\begin{pmatrix} -1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ $=\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ So $x=4$ and $y=-2$	[3]
Question 4 [2011 EOY, Q5]	marks
Given that $\begin{pmatrix} 7 & -1 & -1 \\ -3 & m & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, show that $m=1$. **Solution:** $(-3)(1)+m(4)+(0)(3)=1$ $\Rightarrow m=1$	[2]

Hence solve the following simultaneous equations. $$\begin{aligned} x+\quad y+\quad z&=1\\ 3x+4y+3z&=2\\ 3x+3y+4z&=-3 \end{aligned}$$ **Solution:** $$\begin{pmatrix}1&1&1\\3&4&3\\3&3&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7&-1&-1\\-3&1&0\\-3&0&1\end{pmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix}$$ $$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}8\\-1\\-6\end{pmatrix}$$	[3]
Question 5 [2012 EOY, Q1]	marks
Write down the inverse of the matrix $\begin{pmatrix}-2&7\\3&-10\end{pmatrix}$ and use this to solve the simultaneous equations $7y-2x-8=0,$ $3x-10y+11=0.$ **Solution:** $$\begin{pmatrix}-2&7\\3&-10\end{pmatrix}^{-1}=\frac{1}{(20)-(21)}\begin{pmatrix}-10&-7\\-3&-2\end{pmatrix}$$ $$=\begin{pmatrix}10&7\\3&2\end{pmatrix}$$ $$\begin{pmatrix}-2&7\\3&-10\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\-11\end{pmatrix}$$ $$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10&7\\3&2\end{pmatrix}\begin{pmatrix}8\\-11\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$	[4]

Question 6 [2013 EOY, Q3]	marks
(i) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, the inverses of the matrices $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$. **<u>Solution:</u>** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1.5 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$	[2]
(ii) Use $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ from **(i),** to find the matrix $\mathbf{M}$ $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \mathbf{M} \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}$. **<u>Solution:</u>** $\mathbf{M} = \begin{bmatrix} 1 & -1.5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$ $= \begin{bmatrix} -5 & -3 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$ $= \begin{bmatrix} -2 & 5 \\ 4 & -12 \end{bmatrix}$	[3]

Question 7 [2014 EOY, Q4]	marks
Two outlets of a local bakery sold the following types of cakes on a particular day.	

	Pandan cake	Chocolate cake
Outlet *A*	100	120
Outlet *B*	70	85

The costs of baking a pandan cake and a chocolate cake are \$5 and \$6 respectively. In both outlets, each pandan cake was sold at \$$x$, and each chocolate cake was sold at \$$y$.

It is given that $\mathbf{P} = \begin{pmatrix} 100 & 120 \\ 70 & 85 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(i)
Write down an expression connecting **P**, **Q** and **R**, such that it represents the profit earned by each outlet on that particular day. [1]

Solution:

$$\mathbf{P(R-Q)} \text{ or } \begin{pmatrix} 100 & 120 \\ 70 & 85 \end{pmatrix}\begin{pmatrix} x-5 \\ y-6 \end{pmatrix} \text{ or } \begin{pmatrix} 100 & 120 \\ 70 & 85 \end{pmatrix}\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right]$$

(ii)
If the profits earned by Outlet *A* and Outlet *B* on that particular day were \$1710 and \$1205 respectively, find the selling price for each type of cake using the inverse matrix method. [4]

Solution:

$$\begin{pmatrix} 100 & 120 \\ 70 & 85 \end{pmatrix}\begin{pmatrix} x-5 \\ y-6 \end{pmatrix} = \begin{pmatrix} 1710 \\ 1205 \end{pmatrix}$$

$$\begin{pmatrix} x-5 \\ y-6 \end{pmatrix} = \begin{pmatrix} 100 & 120 \\ 70 & 85 \end{pmatrix}^{-1}\begin{pmatrix} 1710 \\ 1205 \end{pmatrix}$$

$$\begin{pmatrix} x-5 \\ y-6 \end{pmatrix} = \frac{1}{100}\begin{pmatrix} 85 & -120 \\ -70 & 100 \end{pmatrix}\begin{pmatrix} 1710 \\ 1205 \end{pmatrix}$$

$$\begin{pmatrix} x-5 \\ y-6 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 8 \end{pmatrix}$$

$\therefore x = 12.5, y = 14$

The price of pandan cake is \$12.50 each, and the price of chocolate cake is \$14 each.

Question 8 [2015 EOY, Q7]	marks
1 A factory makes tables and chairs. The following table shows the breakdown of the amount of time taken, as well as the amount of wood and paint needed to produce each item.	

	Time taken (hours)	Wood (number of blocks)	Paint (number of tins)
Table	6	4	3
Chair	3	3	2

Labour costs \$10 per hour, while wood costs \$3 per block and each tin of paint costs \$5.

It is given that $\mathbf{A} = \begin{pmatrix} 6 & 4 & 3 \\ 3 & 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 10 \\ 3 \\ 5 \end{pmatrix}$ and $\mathbf{C} = \mathbf{AB}$.

(i)

Evaluate **C** and explain what the numbers in **C** represent. [2]

Solution:

$\mathbf{C} = \begin{pmatrix} 87 \\ 49 \end{pmatrix}$ the numbers represent the cost of producing one table and one chair (\$87 and \$49 respectively).

(ii)

If the factory makes 20 tables and 50 chairs, represent the information as a row matrix **D**. Evaluate **DC** and explain what the answer represents. [3]

Solution:

$\mathbf{D} = (20 \quad 50)$ $\mathbf{DC} = (4190)$, it represents the total cost (\$4190) of producing 20 tables and 50 chairs.