NANYANG JUNIOR COLLEGE

JC2 Common Test 2

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **26th June 2023**

3 Hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages.

1 Given that **u**, **v** and **w** are linearly independent vectors, show that the vectors

$$\mathbf{u}+\mathbf{v}-\mathbf{w},\ \mathbf{v}+\mathbf{w}-\mathbf{u},\ \mathbf{w}+\mathbf{u}-\mathbf{v}$$

are linearly independent. [2]

2 Use the substitution $z=\frac{\mathrm{d}y}{\mathrm{d}x}-y$ to solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}-3y=2\mathrm{e}^x,$$

given that $y=1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2$ when $x=0$. [7]

3 The mapping $\mathrm{F}:\mathbf{M}_{2\times 2}(\mathbb{R})\to\mathbb{R}$ is given by

$$\mathrm{F}\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right]=\int_{-1}^{1} px^2+(q+r)x+s\ \mathrm{d}x.$$

(a) Show that F is a linear transformation. [3]
(b) Find a basis for the null space of F. [3]
(c) Hence, state the range space of F. [1]

A matrix **A** is said to be antisymmetric if $\mathbf{A}=-\mathbf{A}^T$. Let W denote the set of antisymmetric matrices in $\mathbf{M}_{2\times 2}(\mathbb{R})$.

(d) Show that W is a subspace of the null space of F. State the dimension of W. [4]

4 Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+2y=x\mathrm{e}^{3x}$ with the initial conditions $y=0$ when $x=0$.

(a) Use Euler's method with step size 0.5 to approximate the value for y when $x=1$, leaving your answer in 3 decimal places. [2]
(b) Use the Improved Euler's method with step size 1 to approximate the value for y when $x=1$, your answer in 3 decimal places. [2]

It is given that the exact solution to the differential equation is $y=\frac{1}{5}x\mathrm{e}^{3x}-\frac{1}{25}\mathrm{e}^{3x}+\frac{1}{25}\mathrm{e}^{-2x}$.

(c) Use an analytical method to explain if the Euler Method gives an under-estimate or over-estimate of the true value of y when $x=1$. [3]
(d) Determine which of the methods in **(a)** and **(b)** gives a better approximation. [2]

5 A garden landscaper is designing a canopy for a rectangular plot of land to make it aerodynamic and efficient at shedding rainwater. The landscaper models the canopy with the following equation

$$z = \mathrm{f}(x, y), \text{where } \mathrm{f}(x, y) = 7 - 0.05y^2 - 3x^{2/3},$$

with domain $D = \{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1, -2 \le y \le 2\}$ representing the land; x, y and z are measured in appropriate units.

It is assumed that raindrops will follow the path of steepest descent.

(a) Write down $\nabla \mathrm{f}(x, y)$, and hence determine the points on the canopy for which raindrops landing on do not necessarily follow the initial path of descent prescribed by the assumption. [2]

(b) A surface is considered efficient in shedding rainwater if the rate of descent of a raindrop at any point on the surface is at least 3, that is, the height of a raindrop decreases instantaneously by at least 3 units for 1 unit of displacement from the point.

(i) Find an expression for the rate of descent of a raindrop at $(x, y, \mathrm{f}(x, y))$ under the given assumption. [1]

(ii) Comment with justification, whether the canopy is considered efficient in shedding rainwater. [1]

The landscaper plans to support the canopy with steel beams. Two vertical steel beams are to be erected on the land at $(-a, b, 0)$ and $(a, b, 0)$ to the canopy at P and Q respectively. Another steel beam supports the canopy by joining P and Q along the vertical trace of $z = \mathrm{f}(x, y)$ from P to Q. The total length of the three beams is denoted by $L(a, b)$.

(c) Ignoring thickness of the beams, show that

$$L(a, b) = 2\mathrm{f}(a, b) + 2\int_0^a \frac{\sqrt{x^{2/3} + 4}}{x^{1/3}} \, \mathrm{d}x. \qquad [5]$$

(i) Evaluate $L(a, b)$ in terms of a and b. Deduce that for a fixed value of a, the largest possible value of $L(a, b)$ is $L(a, 0)$. [3]

(ii) Hence determine the largest possible value of $L(a, b)$. [1]

6 **(i)** Describe the geometrical relationship between the points on the Argand diagram representing the complex numbers z and $z\mathrm{e}^{\mathrm{i}\theta}$. [1]

(ii) The distinct complex numbers α, β and γ are represented by the points A, B and C respectively.
Given that A, B and C are the vertices of an equilateral triangle taken in an anticlockwise order, state the value of $\arg\left(\dfrac{\gamma-\alpha}{\beta-\alpha}\right)$ and write down the condition that α, β and γ must satisfy. [2]

(iii) Deduce from **(ii)** or show otherwise that the complex number

$$\frac{\gamma-\alpha}{\beta-\alpha}+\frac{\beta-\alpha}{\gamma-\alpha}$$

is purely real and give its value. [2]

(iv) Hence show that $\dfrac{\gamma-\alpha}{\beta-\alpha}$ and $\dfrac{\beta-\alpha}{\gamma-\alpha}$ are the roots of the quadratic equation

$$z^2-z+1=0$$

and find their values in the form $a+b\mathrm{i}$ where a and b are real numbers. [3]

(v) The complex numbers α', β' and γ' are represented by the points A', B' and C' respectively. Given that A', B' and C' are respectively the reflections of A, B and C in the imaginary axis, find, in exponential form, the value of

$$\frac{\gamma'-\alpha'}{\beta'-\alpha'}.$$ [2]

7 Consider the equation $\mathrm{e}^{-x}+\ln\left(\dfrac{x}{5}\right)=0$.

(a) Show that there is exactly one real root. [3]

(b) Given that the root of the equation lies in the interval $[1,5]$, use linear interpolation to find a first approximation of the root, giving your answer to 5 decimal places. [2]

(c) Using the Newton-Rhapson method with your answer in **(b)** as the initial approximation, find the root to 3 decimal places. [2]

(d) The Newton-Rhapson method fails when the initial approximation takes a value greater than or equal to k. Find k. [3]

8 Guppies are well known for their high reproduction rate. In a research study, an ichthyologist, a fish scientist, wishes to study the reproduction of guppies when they are given a special diet. He believes that with this special diet, the number of guppies, G_n , is dependent on the interactions between the number of male guppies, M_n, and female guppies, F_n, in n months. He hypothesised that this can be modelled by

$$M_{n+2} - G_{n+1} = 2M_n - F_{n+1},$$
$$F_{n+1} = 2G_n - 2M_n,$$

where $n \geq 0$. Initially, the research starts off with 10 female and 2 male guppies. After a month, there are 4 male guppies.

(a) Find a first order recurrence relation for F_n and a second order recurrence relation for M_n. Hence solve the recurrence relations and find a solution for G_n. [8]

(b) Unfortunately, at the end of the 3rd month, there was a sudden spike in ammonia. This toxicity damaged the gills and internal organs of guppies. This led to the death of 75% of the population of male guppies and 50% of the population of female guppies. The next 3 months saw no growth in population. Subsequently, the ichthyologist assumes that population growth would revert back to normalcy and there would be 5 male guppies in the 7th month. Find the projected population of guppies at the end of the year. [4]

9 In 2024, the comet C/2023 A3 will travel near planet Earth. Its path is modelled by the curve C_1.
The curve C_1 has polar equation

$$C_1 : r = \frac{5}{2 - 3\cos\theta}.$$

The comet travels on the part of C_1 where $r\cos\theta \geq -1$.

(a) Find the cartesian equations of the asymptotes of C_1. [3]

(b) A satellite is located at the centre of C_1. Find the cartesian coordinates of the satellite. Hence, or otherwise, find the shortest distance from the satellite to the comet. [2]

The region R is enclosed by C_1 and the line, L with cartesian equation $y = \sqrt{3}x$.

(c) Find the polar coordinates of the points of intersection of L and C_1. Hence write down a definite integral for the area of R and use Simpson's rule with 4 strips to find an approximate value. [5]

10 **(a)** The matrix $\mathbf{M}$ is such that $\det(\mathbf{M}) \neq 0$ and has a non-zero eigenvalue λ and associated eigenvector $\mathbf{x}$. Find an eigenvalue and an associated eigenvector for $\mathbf{M}^{-1}$. [2]

(b) Given that the matrix $\mathbf{M}_1$ has the same eigenvector as $\mathbf{M}$ corresponding to the eigenvalue μ, find an eigenvalue and eigenvector for $\mathbf{M}\mathbf{M}_1$. [2]

The nullspace of matrix $\mathbf{A}$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

(c) State, with a reason, the value of an eigenvalue of $\mathbf{A}^T\mathbf{A}$ and a corresponding eigenvector. [1]

(d) Hence, find an eigenvalue and a corresponding eigenvector to the matrix

$$\mathbf{B} = \left(3\mathbf{I} - \mathbf{A}^T\mathbf{A}\right)\left(3\mathbf{I} + \mathbf{A}^T\mathbf{A}\right)^{-1}.$$ [5]

(e) If the remaining eigenvalues of $\mathbf{A}$ are positive, show that all other eigenvalues γ of $\mathbf{B}$ must satisfy the inequality $-1 < \gamma < 1$. [5]

——END OF PAPER——