RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

Tutorial S2B: Binomial Distribution

Section A (Discussion Questions)

1 State, with a reason, whether a binomial distribution could be used in each of the following problems. If binomial distribution is an acceptable model, define the random variable clearly and state its parameters.

(a) A gambler has a biased coin for which the probability of obtaining a head in any toss is 0.56. Find the probability of getting 6 heads if he tosses the coin 8 times.

(b) A fair coin is spun until a head occurs. Find the probability that 8 spins are necessary, including the one on which the head occurs.

(c) A jar contains 49 balls numbered 1 to 49. Six of the balls are drawn at random without replacement. Find the probability that four out of the six balls drawn have an even score.

Solution:

(a) Since there are 8 independent trials (tosses of coins), each toss either yields a head or tail and the probability of getting a head stays constant at 0.56 from toss to toss, binomial distribution is an acceptable model.

Let X be the no. of heads obtained, out of 8 tosses.

$X \sim \mathrm{B}(8, 0.56)$

$\mathrm{P}(X=6) = \binom{8}{6}(0.56)^6(0.44)^2 = 0.167$

(b) No. For binomial distribution to be used, it should consist of n independent spins but in this case, there are no fixed n number of spins.

(c) No. For binomial distribution to be used, the draws should be independent of each other, which is not the case here since the balls are drawn with no replacement.

Note: Probability of drawing an even-numbered ball is <u>constant</u> regardless of the draw.

Eg P(drawing an even-numbered ball in 1st draw) = $\frac{24}{49}$

P(drawing an even-numbered ball in 2nd draw)

= P(drawing an even-numbered ball in 2nd draw and odd-numbered ball in 1st draw)

+ P(drawing an even-numbered ball in 2nd draw and even-numbered ball in 1st draw)

$= \frac{25}{49} \times \frac{24}{48} + \frac{24}{49} \times \frac{23}{48} = \frac{24}{49}$

Similarly, P(drawing an even-numbered ball in 3rd draw)

=… = P(drawing an even-numbered ball in 6th draw) = $\frac{24}{49}$

2 Given $X \sim \mathrm{B}(10, 0.2)$, find

(i) $\mathrm{P}(X = 0)$; **(ii)** $\mathrm{P}(X \le 2)$; **(iii)** $\mathrm{P}(X \ge 5)$;

(iv) $\mathrm{P}(4 \le X \le 8)$; **(v)** $\mathrm{P}(2 \le X < 6)$; **(vi)** $\mathrm{E}(X)$;

(vii) $\mathrm{Var}(X)$.

[**(i)** 0.107 **(ii)** 0.678 **(iii)** 0.0328 **(iv)** 0.121 **(v)** 0.618 **(vi)** 2 **(vii)** 1.6]

(i) and **(ii)** involves direct application of graphing calculator.

(iii) $\mathrm{P}(X \ge 5) = 1 - \mathrm{P}(X \le 4)$
$= 0.0328$

(iv) $\mathrm{P}(4 \le X \le 8) = \mathrm{P}(X \le 8) - \mathrm{P}(X \le 3)$
$= 0.121$

(v) $\mathrm{P}(2 \le X < 6) = \mathrm{P}(X \le 5) - \mathrm{P}(X \le 1)$
$= 0.618$

(vi) $\mathrm{E}(X) = np$
$= (10)(0.2)$
$= 2$

(vii) $\mathrm{Var}(X) = np(1-p)$
$= (10)(0.2)(1-0.2)$
$= 1.6$

3 A die is biased such that the probability of getting a six is 0.18. If the die is thrown ten times, find

(i) the probability that the number of sixes that turn up is odd,

(ii) the expectation and variance of the number of sixes that turn up.

[(i) 0.494, **(ii)** 1.8, 1.476**]**

(i)

Let X be the number of sixes obtained, out of 10 throws. $X \sim \mathrm{B}(10, 0.18)$

$\mathrm{P}(X \text{ is odd}) = \mathrm{P}(X=1) + \mathrm{P}(X=3) + \mathrm{P}(X=5) + \mathrm{P}(X=7) + \mathrm{P}(X=9)$
$= 0.494 \ (3 \text{ s.f.})$

(ii)

$\mathrm{E}(X) = np$
$= 10(0.18)$
$= 1.8$

$\mathrm{Var}(X) = np(1-p)$
$= 10(0.18)(1-0.18)$
$= 1.476$

Expectation and variance of number of sixes that turn up is 1.8 and 1.476 respectively.

4 Two fair dice are thrown. Find the probability that the total score is two-figured.
If five people were each to throw two dice, find the probability that exactly three of them will get a two-figured total score.

[$\frac{1}{6}$, 0.0322**]**

Total score is two-figured : $(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)$.

Probability that total score is two-figured = $\frac{6}{36}=\frac{1}{6}$

Let Y be the number of people, out of the 5, who get a two-figured total score. $Y \sim \mathrm{B}\left(5,\frac{1}{6}\right)$

$\mathrm{P}(Y=3)=0.0322\ (3\text{ s.f.})$

5 **RJC Prelim 1997/02/Q7(a)(modified)**
The probability that a patient suffering from a particular disease will be cured following a new treatment is 0.92.

(i) If the treatment is given to 10 patients, show that the probability that at least 9 out of the 10 patients will be cured is 0.812, correct to 3 significant figures.

(ii) This treatment is being tested at 12 hospitals where in each hospital, the treatment is given to 10 patients. Find the probability that more than 10 of the 12 hospitals will have a success rate of at least ninety percent.

[(ii) 0.310**]**

(i)
Let X be the number of patients, out of 10, who will be cured following the new treatment.

$X \sim \mathrm{B}(10, 0.92)$

$\mathrm{P}(X \geq 9)=\mathrm{P}(X=9)+\mathrm{P}(X=10)$

$= 0.81212 \quad = 0.812\ (3\text{ s.f.})\ (\text{Shown})$

OR:

$\mathrm{P}(X \geq 9)=1-\mathrm{P}(X \leq 8)$

$= 1-0.18788$

$= 0.81212 \quad = 0.812\ (3\text{ s.f.})\ (\text{Shown})$

(ii)
Let Y be the number of hospitals each with 10 patients, out of 12 hospitals, which will have a success rate of at least 90 percent.
Probability of a hospital having success rate of at least 90 percent = 0.812 (from (i))

$Y \sim \mathrm{B}(12, 0.812)$

$\mathrm{P}(Y>10)=1-\mathrm{P}(Y \leq 10)$

$= 0.310\ (3\text{ s.f.})$

6 **SRJC Prelim 2007/02/Q10a**

According to the school rules, a student who arrives at school after 0730 hours is considered late.

(i) During the school term, a boy cycles to school on 5 days each week. On any given day, the probability that he arrives after 0730 hours is 0.1. For a period of 4 weeks, calculate the probability that he is late on at least one day but not more than 3 days.

When the boy has cycled to school for the *n*th time, the probability that he arrived at school late at least once is greater than 0.99. Calculate the least value of *n*.

(ii) A girl travels to school by bus on 5 days a week. Over a long period of time, the variance of the number of days per week on which she is late is 0.8. Given that *p* denotes the probability that she arrives late and that $p < 0.5$, evaluate *p*.

[(i) 0.745, 44 **(ii)** 0.2**]**

Let X be the number of days a student is late in 4 weeks.

$$X \sim \mathrm{B}(20, 0.1)$$

$\mathrm{P}(1 \le X \le 3) = \mathrm{P}(X \le 3) - \mathrm{P}(X = 0)$

$= 0.745$

Let Y be the number of days a student is late in n days.

$Y \sim \mathrm{B}(n, 0.1)$

$\mathrm{P}(Y \ge 1) > 0.99$

$\Rightarrow 1 - \mathrm{P}(Y = 0) > 0.99$

$\Rightarrow \mathrm{P}(Y = 0) < 0.01$

$\Rightarrow \binom{n}{0}(0.1)^0(0.9)^n < 0.01$

$\Rightarrow n\ln(0.9) < \ln(0.01)$

$\Rightarrow n > \dfrac{\ln(0.01)}{\ln(0.9)} = 43.70869$

least value of n is 44.

Alternatively using GC (table),

$\mathrm{P}\left(Y = 0\right) \le 0.01$

n	$\mathrm{P}\left(Y = 0\right)$
43	$0.0108 > 0.01$
44	$0.00970 < 0.01$

least value of n is 44.

Let W be the number of days the girl is late in a week.

$W \sim \mathrm{B}(5, p)$

Variance = 0.8

$\Rightarrow 5p(1 - p) = 0.8$

$\Rightarrow p^2 - p + 0.16 = 0$

$\Rightarrow p = 0.8$ or 0.2

Since $p < 0.5$, $p = 0.2$

7 A female doctor wishes to test a new treatment for a certain disease. She asks patients who come to her clinic suffering from this disease if they are willing to take part in a trial. Based on past experiences, she finds that the probability of such a patient willing to take part is 0.3.

(i) Show that the probability that at least 2 patients are willing to take part in this trial when she has asked 8 patients is 0.745, correct to 3 significant figures. What assumption do you have to make to justify the use of the distribution in your working?

(ii) The doctor decides to ask n patients. Find the least number of patients she has to ask so that she will have a probability of more than 0.8 that at least 2 patients are willing to take part in this trial.

(iii) Find the most likely number of patients who are willing to take part in this trial when she asks 9 patients.

(iv) State the expected number of patients who are willing to take part in this trial when she asks 9 patients.

[(ii) 9 **(iii)** 2 and 3 **(iv)** 2.7**]**

(i) Let X be the number of patients, out of 8 that she asked, who are willing to take part in a trial.
Assumption: The event of a patient's decision to take part in the trial is independent of the event of another patient's.

$X \sim \mathrm{B}(8, 0.3)$

$$
\begin{aligned}
\mathrm{P}(X \geq 2) &= 1 - \mathrm{P}(X \leq 1) \\
&= 1 - 0.25530 \\
&= 0.74470 \\
&= 0.745 \ (3 \text{ s.f.}) (\text{Shown})
\end{aligned}
$$

(ii) Let Y be the number of patients who are willing to take part in the trial, out of n patients.

$Y \sim \mathrm{B}(n, 0.3)$

$\mathrm{P}(Y \geq 2) > 0.8$

$1 - \mathrm{P}(Y \leq 1) > 0.8$

$\mathrm{P}(Y \leq 1) < 0.2$

From GC,

n	$\mathrm{P}(Y \leq 1)$
8	$0.25530 > 0.2$
9	$0.19600 < 0.2$

$\therefore$ least $n = 9$

(iii) When $n = 9,\ Y \sim \mathrm{B}(9, 0.3)$

From GC,

$\mathrm{P}(Y=1) = 0.15565$

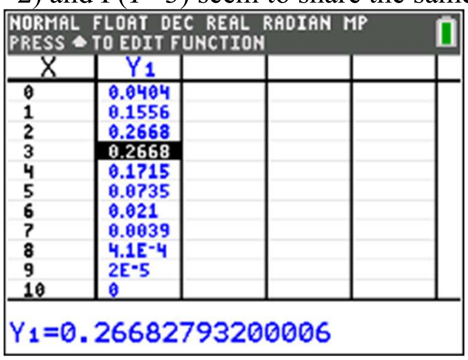
$\mathrm{P}(Y=2) = 0.26683$

$\mathrm{P}(Y=3) = 0.26683$

$\mathrm{P}(Y=4) = 0.17153$

$\therefore$ most likely number = 2 and 3

Note:
From GC, P(Y=2) and P(Y=3) seem to share the same value.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X	Y1
0	0.0404
1	0.1556
2	0.2668
3	0.2668
4	0.1715
5	0.0735
6	0.021
7	0.0039
8	4.1E-4
9	2E-5
10	0

Y1=0.26682793200006

We can verify this by using the pdf formulae.

P(Y=2) $= {}^9C_2(0.3)^2(0.7)^7 = 36(0.3)^2(0.7)^7$

P(Y=3) $= {}^9C_3(0.3)^3(0.7)^6 = 84(0.3)^3(0.7)^6 = 84\left[\dfrac{0.3}{0.7}\right](0.3)^2(0.7)^7 = 36(0.3)^2(0.7)^7$

Note: As y increases, P(Y=y) increases to a max and then decreases (lecture notes pg 4 Section 1.4)

(iv)

$\mathrm{E}(Y) = np$

$= 9(0.3)$

$= 2.7$

8 HCI Prelim 2009/02/Q8 (modified)

In a promotional campaign organized by the management of the petrol filling station, a customer who refills his car with at least \$50 worth of petrol is entitled to draw 5 tickets from a lucky draw box at random. $\frac{1}{6}$ of the tickets in the lucky draw box are winning tickets while the rest are non-winning tickets. The 5 tickets drawn by a customer are put back into the box before the draw by the next customer.

State an assumption needed for a binomial distribution to be a valid model for the number of winning tickets drawn in a draw of 5 tickets.

Using a binomial distribution as the valid model, find the probability that among 3 randomly chosen customers who participated exactly once in a draw, only one of them obtained at least one winning ticket in the draw of 5 tickets.

[0.290**]**

Assumption: The total no. of tickets in the box has to be large so that the drawing of 5 tickets without replacement is approximately independent.

For illustration purposes of the above assumption:

Let n be the total number of tickets in the box.

n	P (2nd ticket drawn is winning | 1st ticket drawn is winning) [**without** replacement]	P (2nd ticket drawn is winning | 1st ticket drawn is winning) [**with** replacement]
60	$\frac{9}{59} = 0.15254$ (5 s.f.)	$\frac{10}{60} = \frac{1}{6} = 0.16667$ (5 s.f.)
6000	$\frac{999}{5999} = 0.16653$ (5 s.f.)	$\frac{1000}{6000} = \frac{1}{6} = 0.16667$ (5 s.f.)

Possible wrong answers:

(1) Assume that the 5 tickets are drawn one by one with replacement. [This would have changed the original lucky draw mechanism.]

(2) Assume that the probability of drawing a winning ticket is constant for each draw. Even without replacement, this probability is in fact constant.
Illustration using no. of tickets in the box = 60:
P (2^{nd} ticket drawn is winning)

$= \text{P (1}^{\text{st}}\text{ ticket is winning)} \times \text{P(2}^{\text{nd}}\text{ ticket is winning | 1}^{\text{st}}\text{ ticket is winning)}$

$+$

$\text{P (1}^{\text{st}}\text{ ticket is not winning)} \times \text{P(2}^{\text{nd}}\text{ ticket is winning | 1}^{\text{st}}\text{ ticket is not winning)}$

$= \frac{10}{60} \times \frac{9}{59} + \frac{50}{60} \times \frac{10}{59}$

$= \frac{1}{6}$

Let X be the number of winning tickets in a draw of 5 tickets.

$X \sim \text{B}(5, \frac{1}{6})$

$\text{P}(X \geq 1) = 1 - \text{P}(X = 0) = 0.59812$ (5sf)

<u>Method 1:</u>

Required probability $= 3 \times \text{P}(X \geq 1)[\text{P}(X = 0)]^2$

$= 3 \times (0.59812)(1 - 0.59812)^2$

≈ 0.290

<u>Method 2:</u>

Let Y be the number of customers out of 3 with at least one winning ticket.

$Y \sim \text{B}(3, 0.59812)$

Required probability $= \text{P}(Y = 1) \approx 0.290$

9 **NJC Prelim 9758/2021/02/Q8**

National Fruit Company owns a large tomato farm. The tomatoes produced are harvested and sold in boxes of 25. It is known that $100p\%$ of the tomatoes are rotten. For these boxes, the mean number of rotten tomatoes in a box is 1.

(i) Explain why the context above may not be well-modelled by a binomial distribution. [1]

Assume now that the context above is well-modelled by a binomial distribution.

(ii) State the value of p. [1]

(iii) Find the probability that a box chosen at random has less than 2 rotten tomatoes. [2]

(iv) A customer chose a box and inspected the contents individually. Find the probability that the twenty-first tomato is the fourth rotten tomato and no rotten tomatoes are found subsequently. [3]

Boxes that contains at least 24 tomatoes that are not rotten are deemed satisfactory.

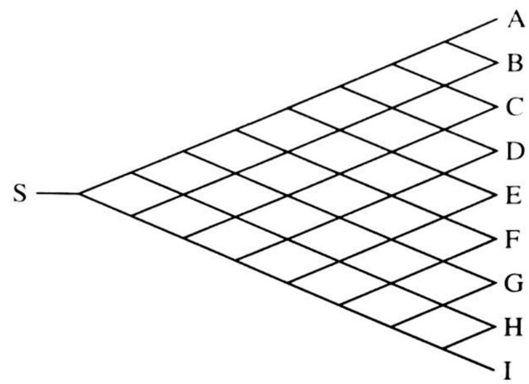
(v) A customer first picks 3 boxes of tomatoes, of which at least 2 boxes are satisfactory. The customer then decides to buy another 5 boxes. Find the probability that exactly 6 of the 8 boxes are satisfactory. [4]

[(ii) 0.04 (iii) 0.736 (iv) 0.00124 (v) 0.335**]**

(i)	The **probability** that a tomato is rotten **may not be the same** for all tomatoes because the tomatoes from the farm may be subjected to different treatment, thereby affecting the quality of the tomatoes. OR The event that a tomato is rotten may not be **independent** of another tomato because a rotten tomato may **affect** the quality of other tomatoes from the same plant.
(ii)	$25(p)=1$ $p=\dfrac{1}{25}=0.04$
(iii)	Let X be the number of rotten tomatoes out of 25. $X \sim \mathrm{B}(25, 0.04)$ $\mathrm{P}(X<2)=\mathrm{P}(X \le 1)$ $=0.736\,(3\text{ s.f.})$
(iv)	Let Y be the number of rotten tomatoes out of 20. $Y \sim \mathrm{B}(20, 0.04)$ Required probability $=\mathrm{P}(Y=3)(0.04)(0.96)^4$ $=(0.036450)(0.04)(0.96)^4$ $=0.00124$ (3 s.f.)

(v)	Let Q be the number of satisfactory boxes out of 3. $Q \sim \mathrm{B}(3, 0.73581)$ Let R be the number of satisfactory boxes out of 5 $R \sim \mathrm{B}(5, 0.73581)$ $\mathrm{P}(\text{exactly 6 boxes satisfactory} \mid Q \geq 2)$ $= \dfrac{\mathrm{P}(\text{exactly 6 boxes satisfactory and } Q \geq 2)}{\mathrm{P}(Q \geq 2)}$ $= \dfrac{\mathrm{P}(Q=2)\mathrm{P}(R=4)+\mathrm{P}(Q=3)\mathrm{P}(R=3)}{\mathrm{P}(Q \geq 2)}$ $= \dfrac{\mathrm{P}(Q=2)\mathrm{P}(R=4)+\mathrm{P}(Q=3)\mathrm{P}(R=3)}{1-\mathrm{P}(Q \leq 1)}$ $= 0.335 \text{ (3 s.f.)}$

10 9758/2018/02/Q6



In a computer game. a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A – I (see diagram).

(i) Show that the probability that the bug finishes its journey at D is $56p^5q^3$. [2]

(ii) Given that the probability that the bug finishes its journey at D is greater than the probability that the bug finishes its journey at any one of the other endpoints, find exactly the possible range of values of p. [4]

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(iii) Find the probability that, in this version of the game, the bug reaches one of the endpoints A – I, without being swallowed up by a black hole. [1]

[(ii) $\dfrac{5}{9} < p < \dfrac{2}{3}$ (iii) 0.430]

(i)	Note that for any path allowing the bug to move from S to D, the bug has to take 5 left forks and 3 right forks in any order. The required probability is ${}^8C_5 p^5 q^3 = 56p^5q^3$ (shown).
(ii)	This is a binomial distribution in disguise. Let X denote the number of left forks out of 8. (the rest will be right forks) $X \sim \mathrm{B}(8, p)$ We want ${}^8C_5 p^5 q^3 = \mathrm{P}(X=5)$ to be the largest, so $\mathrm{P}(X=0) \le \mathrm{P}(X=1) \ldots \le \mathrm{P}(X=4) < \mathrm{P}(X=5) > \mathrm{P}(X=6) \ge \mathrm{P}(X=7) \ge \mathrm{P}(X=8)$ $\mathrm{P}(X=4) < \mathrm{P}(X=5)$ and $\mathrm{P}(X=5) > \mathrm{P}(X=6)$ ${}^8C_4 p^4 q^4 < {}^8C_5 p^5 q^3$ ${}^8C_5 p^5 q^3 > {}^8C_6 p^6 q^2$ $70q < 56p$ $56q > 28p$ $70(1-p) < 56p$ $56(1-p) > 28p$ $p > \frac{5}{9}$ $p < \frac{2}{3}$ Thus, $\frac{5}{9} < p < \frac{2}{3}$
(iii)	Required probability is $(1-0.1)^8 = 0.9^8 = 0.430$ (3 s.f.)

THE END