RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 6B: Applications of Differentiation

SYLLABUS INCLUDES

- Problems involving tangents and normal to curves, including cases where the curve is defined implicitly or parametrically
- Local maxima and minima problems
- Connected rates of change problems

CONTENT

1 Tangents and Normals

Recall that equation of a (straight) line with gradient m passing through $\left(x_1,\, y_1\right)$ is

$$y - y_1 = m\,(x - x_1)$$

Let $P\left(k,\ \mathrm{f}(k)\right)$ be a point on the graph of $y = \mathrm{f}(x)$. In this case,

- gradient of the tangent to the curve at P is $\mathrm{f}'(k)$.
  So equation of the **tangent to the curve at** P is

$$\boxed{y - \mathrm{f}(k) = \mathrm{f}'(k)\left(x - k\right)}$$

- gradient of the normal to the curve at P is $-\dfrac{1}{\mathrm{f}'(k)}$, provided $\mathrm{f}'(k) \neq 0$.
  So equation of the **normal to the curve at** P is

$$\boxed{y - \mathrm{f}(k) = -\frac{1}{\mathrm{f}'(k)}(x - k)\text{, } \textbf{provided } \mathrm{f}'(k) \neq 0}$$

Qn : What are the equations of the tangent and normal to the curve at point P if $\mathrm{f}'(k) = 0$?

Equation of tangent: $y = \mathrm{f}(k)$ i.e. tangent is horizontal
Equation of normal: $x = k$ i.e. normal is vertical

Example 1

A curve has equation given by $x^2 + 2xy - 2y^2 = -11$. Find

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$,

(ii) the equations of the tangent and normal to the curve at the point (1,3),

(iii) the coordinates of the points on the curve where the tangent is parallel to the x-axis.

Solution

(i) $x^2 + 2xy - 2y^2 = -11$

Differentiate with respect to x,

$$2x + 2y + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(4y - 2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{2y - x}$$

(ii) At (1,3), $\dfrac{dy}{dx}=\dfrac{1+3}{2(3)-1}=\dfrac{4}{5}$

Equation of the tangent to curve at (1,3) is $y-3=\dfrac{4}{5}(x-1) \quad\Rightarrow\quad 5y-4x=11$

Equation of the normal to curve at (1,3) is $y-3=-\dfrac{5}{4}(x-1) \quad\Rightarrow\quad 4y+5x=17$

(iii) When tangent is parallel to the x-axis, $\dfrac{dy}{dx}=\dfrac{x+y}{2y-x}=0 \quad\Rightarrow y=-x \quad ---(1)$

Sub. (1) into $x^2+2xy-2y^2=-11$, $x^2-2x^2-2x^2=-11 \quad\Rightarrow x=\pm\sqrt{\dfrac{11}{3}}$.

Coordinates of the points are $\left(\sqrt{\dfrac{11}{3}},-\sqrt{\dfrac{11}{3}}\right)$ and $\left(-\sqrt{\dfrac{11}{3}},\sqrt{\dfrac{11}{3}}\right)$.

Example 2

The parametric equations of a curve are given by $x=\cos t$, $y=\sin t$. Find the equation of the normal to the curve at the point where $t=\dfrac{\pi}{3}$.

Solution

$x=\cos t \quad\Rightarrow\quad \dfrac{dx}{dt}=-\sin t$,

$y=\sin t \quad\Rightarrow\quad \dfrac{dy}{dt}=\cos t$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}=(\cos t)\left(-\dfrac{1}{\sin t}\right)=-\cot t$

When $t=\dfrac{\pi}{3}$, $x=\cos\dfrac{\pi}{3}=\dfrac{1}{2}$,

$y=\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$,

$\dfrac{dy}{dx}=-\cot\dfrac{\pi}{3}=-\dfrac{1}{\sqrt{3}}$

Gradient of normal at the point where $t=\dfrac{\pi}{3}$ is $\sqrt{3}$.

Equation of normal at the point where $t=\dfrac{\pi}{3}$ is

$$y-\frac{\sqrt{3}}{2}=\sqrt{3}\left(x-\frac{1}{2}\right)$$

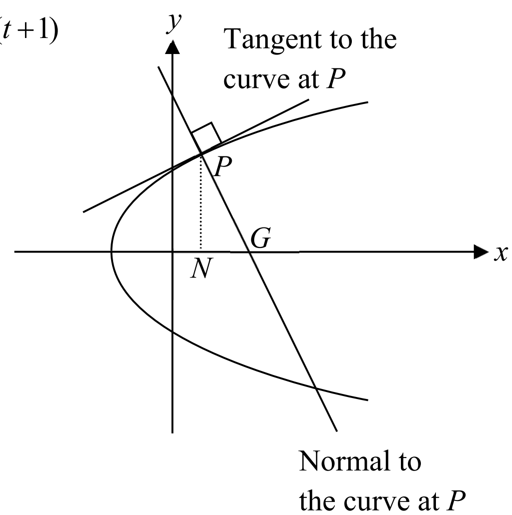
$$\Rightarrow\quad y=\sqrt{3}\,x$$

Example 3

The parametric equations of a curve are given by $x = at(t+2)$, $y = 2a(t+1)$, where a is a positive constant. Find the equation of the normal to the curve at P when $t = p$.

If this normal meets the $x-$axis at G and N is the foot of the perpendicular from P to the $x-$axis, prove that $NG = 2a$.

Solution



$x = at(t+2) = at^2 + 2at \Rightarrow \dfrac{dx}{dt} = 2at + 2a = 2a(t+1)$

$y = 2a(t+1) = 2at + 2a \Rightarrow \dfrac{dy}{dt} = 2a$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{1}{t+1}$

At P, $x = ap(p+2)$, $y = 2a(p+1)$, $\dfrac{dy}{dx} = \dfrac{1}{p+1}$

Equation of normal to the curve at P is

$$y - 2a(p+1) = -\frac{1}{\left(\frac{1}{p+1}\right)}[x - ap(p+2)]$$

$$y = -(p+1)x + ap(p+2)(p+1) + 2a(p+1)$$

$$y = -(p+1)x + a(p+1)(p^2+2p+2)$$

At G, $y = 0$, $x = \dfrac{a(p+1)(p^2+2p+2)}{p+1} = a(p^2+2p+2)$

At N, $x = ap(p+2)$, since $x-$coordinate of N and $x-$coordinate of P are equal.

$\therefore NG = a(p^2+2p+2) - ap(p+2) = 2a$

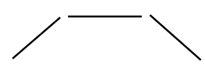
2 Maximization and Minimization

Many real-life situations require that some quantity (e.g. cost of manufacture or fuel consumption) be minimized, i.e. made as small as possible. Other situations require that some quantity (e.g. profit on sales or attendance at a concert) be maximized, i.e. made as large as possible. We can use differentiation to solve many of these problems.

Recall: To determine whether the required quantity y at $x=k$ is minimized or maximized, given that $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=k}=0$.

Method 1 : First Derivative Test

Check the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $x=k^-$ and $x=k^+$.

x	k^- k k^+	k^- k k^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$+$ve 0 $-$ ve	$-$ve 0 $+$ ve
Tangent		
Conclusion	y is maximized at $x=k$	y is minimized at $x=k$

Method 2 : Second Derivative Test

Check the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x=k$.

If $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=k}<0$, then y is maximized at $x=k$

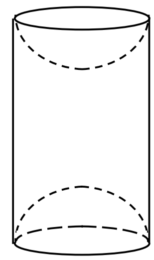
If $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=k}>0$, then y is minimized at $x=k$

Example 4

A solid right circular cylinder has a hemisphere hollowed out from each end. Given that the surface area is $64\pi\ \text{cm}^2$, find the radius of the cylinder in exact form when the volume is a maximum. You need to justify that the volume is a maximum, without the use of the graphing calculator.

Solution

Let h cm and r cm represent the height and radius of the cylinder respectively.



Surface area $= 2\pi rh + 4\pi r^2 = 64\pi$

$$\Rightarrow \quad h = \frac{32-2r^2}{r}$$

Volume of the cylinder, $V = \pi r^2 h - \frac{4}{3}\pi r^3 = \pi r^2\left(\frac{32-2r^2}{r}\right) - \frac{4}{3}\pi r^3 = \pi\left(32r - \frac{10}{3}r^3\right)$

$$\Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}r} = \pi(32-10r^2)$$

When $\frac{\mathrm{d}V}{\mathrm{d}r} = 0 \quad \Rightarrow \quad r^2 = \frac{32}{10} = \frac{16}{5} \quad \Rightarrow \quad r = \frac{4}{\sqrt{5}}$ (since $r > 0$)

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -20\pi r$$

When $r = \frac{4}{\sqrt{5}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -20\pi\left(\frac{4}{\sqrt{5}}\right) = -\frac{80\pi}{\sqrt{5}} < 0$

Alternatively, to show that V is maximum using the 1st derivative test we need to factorise the expression of $\frac{\mathrm{d}V}{\mathrm{d}r}$ and examine the factors that contribute to the sign of $\frac{\mathrm{d}V}{\mathrm{d}r}$ because the question already mentioned V is to be maximized.

So $\frac{\mathrm{d}V}{\mathrm{d}r} = \pi(32-10r^2) = 2\pi\left(16-5r^2\right) = 2\pi\left(4-\sqrt{5}r\right)\left(4+\sqrt{5}r\right)$.

Since $r > 0$, we have $\left(4+\sqrt{5}\,r\right) > 0$. Therefore the only factor that is going to determine the sign of $\frac{\mathrm{d}V}{\mathrm{d}r}$ at $r = \frac{4}{\sqrt{5}}$ is $\left(4-\sqrt{5}\,r\right)$.

r	$\left(\frac{4}{\sqrt{5}}\right)^-$	$\frac{4}{\sqrt{5}}$	$\left(\frac{4}{\sqrt{5}}\right)^+$
$\left(4-\sqrt{5}\,r\right)$	+ve	0	− ve
$\frac{\mathrm{d}V}{\mathrm{d}r}$	+ve	0	− ve

Hence the volume is a maximum when the radius is $\frac{4}{\sqrt{5}}$ cm.

Some guidelines to solve problems involving maximization and minimization :

- Denote each changing quantity by a variable.
- Write down a formula for the quantity to be maximized or minimized.
- From the given or implied information (usually making reference to what is fixed or remains as a constant) in the question, express the quantity to be maximized or minimized in terms of 1 variable only.
- Differentiate and equate derivative to zero for stationary values.
- Justify if quantity is a maximum or minimum (using first or second derivative test).
- Answer the question with a statement.

Remark: 1st derivative test may be easier to conduct on some occasions especially if it is tedious to obtain the 2nd derivative. However, you need to give a complete account for how the sign changes for the 1st derivative and this can be done usually by considering the fully factorised form of the 1st derivative.

Example 5

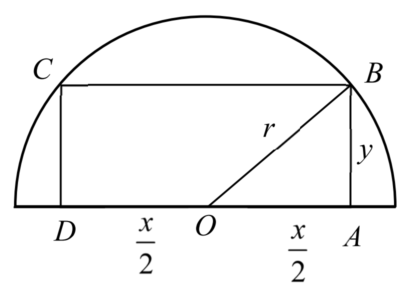
The diagram shows a rectangle $ABCD$ inscribed in a semi-circle with a fixed radius r cm and centre at O. If $AD = x$ cm, find an expression for the perimeter P cm and area A cm^2 of rectangle $ABCD$, in terms of r and x. Show that if A is maximum, then $P = 3\sqrt{2}\,r$.

Solution

From diagram, $\left(\dfrac{x}{2}\right)^2 + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \dfrac{x^2}{4}}$

$P = 2(x+y) = 2\left(x + \sqrt{r^2 - \dfrac{x^2}{4}}\right)$

and $A = xy = x\sqrt{r^2 - \dfrac{x^2}{4}}$



$$\frac{dA}{dx} = \sqrt{r^2 - \frac{x^2}{4}} + \frac{x}{2}\left(r^2 - \frac{x^2}{4}\right)^{-\frac{1}{2}}\left(-\frac{x}{2}\right)$$

$$= \sqrt{r^2 - \frac{x^2}{4}} - \frac{\frac{x^2}{4}}{\sqrt{r^2 - \frac{x^2}{4}}}$$

$$= \frac{r^2 - \frac{x^2}{2}}{\sqrt{r^2 - \frac{x^2}{4}}} = \frac{2r^2 - x^2}{2\sqrt{r^2 - \frac{x^2}{4}}}$$

$$= \frac{\left(\sqrt{2}\,r + x\right)\left(\sqrt{2}\,r - x\right)}{2\sqrt{r^2 - \frac{x^2}{4}}}$$

Alternative (using Implicit Differentiation):

$$A^2 = x^2\left(r^2 - \frac{x^2}{4}\right) = r^2x^2 - \frac{x^4}{4}$$

Differentiate with respect to x,

$$2A\frac{dA}{dx} = 2r^2x - x^3$$

$$\Rightarrow \quad 2A\frac{dA}{dx} = x\left(2r^2 - x^2\right)$$

$$\Rightarrow \quad 2A\frac{dA}{dx} = x\left(\sqrt{2}\,r + x\right)\left(\sqrt{2}\,r - x\right)$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{x\left(\sqrt{2}\,r + x\right)\left(\sqrt{2}\,r - x\right)}{2A}$$

When $\frac{\mathrm{d}A}{\mathrm{d}x}=0 \quad \Rightarrow \quad x=\sqrt{2}\,r$ (since $x>0$)

To show A is maximum, we need to determine the sign of $\frac{\mathrm{d}A}{\mathrm{d}x}$ at $x=\sqrt{2}\,r$ when applying the 1st derivative test.

$$\frac{\mathrm{d}A}{\mathrm{d}x}=\frac{x\left(\sqrt{2}\,r+x\right)\left(\sqrt{2}\,r-x\right)}{2A}$$

Since $x>0$ and $r>0$, we have $\left(\sqrt{2}\,r+x\right)>0$. Therefore the only factor that is going to determine the sign of $\frac{\mathrm{d}A}{\mathrm{d}x}$ at $x=\sqrt{2}\,r$ is $\left(\sqrt{2}\,r-x\right)$

x	$\left(\sqrt{2}\,r\right)^-$	$\sqrt{2}\,r$	$\left(\sqrt{2}\,r\right)^+$
$\left(\sqrt{2}\,r-x\right)$	+ve	0	– ve
$\frac{\mathrm{d}A}{\mathrm{d}x}$	+ve	0	– ve

Note that we need to factorise the expression of $\frac{\mathrm{d}A}{\mathrm{d}x}$ and examine the factors that contribute to the sign of $\frac{\mathrm{d}A}{\mathrm{d}x}$ because the question already mentioned A is to be maximized.

Hence A is maximum when $x=\sqrt{2}\,r$, which gives

$$P=2\left(\sqrt{2}\,r+\sqrt{r^2-\frac{(\sqrt{2}\,r)^2}{4}}\right)=2\left(\sqrt{2}\,r+\sqrt{\frac{r^2}{2}}\right)=2\sqrt{2}\,r+\sqrt{2}\,r=3\sqrt{2}\,r.\text{ (shown)}$$

3 Connected Rates of Change

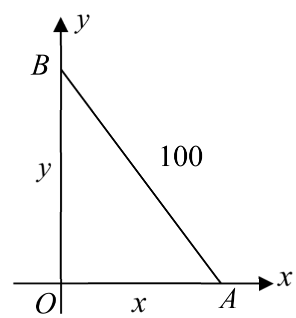
Recall that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the rate of change of y with respect to x. In this section, we deal with rate of change with respect to time, e.g. $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the rate of change of V with respect to t (time).

Example 6

A rod of 100 cm long moves so that its ends A and B remain on the positive $x-$axis and $y-$axis respectively. If A is 80 cm from the origin O and is moving away from the origin at a rate of $20\ \text{cms}^{-1}$, at what rate is the end B coming down?

Solution

Let $OA = x$ cm and $OB = y$ cm.



From diagram,

$$x^2 + y^2 = 100^2.$$

$$\Rightarrow\ y = \sqrt{100^2 - x^2} \quad (\because y > 0)$$

$$\Rightarrow\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{100^2 - x^2}}(-2x) = -\frac{x}{\sqrt{100^2 - x^2}}$$

Now $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}x}{\mathrm{d}t}$

When $x = 80$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 20$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{80}{\sqrt{100^2 - 80^2}} = -\dfrac{80}{60} = -\dfrac{4}{3}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{4}{3} \times 20 = -\frac{80}{3}$$

Hence B is moving down at a rate of $\dfrac{80}{3}\ \text{cms}^{-1}$.

Alternative:

Differentiate $x^2 + y^2 = 100^2$ wrt t,

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

When $x = 80$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 20$ and $y = \sqrt{100^2 - 80^2} = 60$, so $2(80)(20) + 2(60)\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{80}{3}$$

Some guidelines to solve questions involving rate of change :

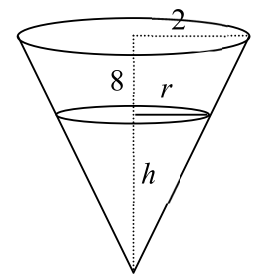
- Denote each changing quantity by a variable.
- Find the equations relating the variables.
- Use the **chain rule** to link up the derivatives.
- Write down the values of the variables and the given rates of change.
- Solve for the unknown rate.
- Answer the question with a statement.

Example 7

A tank is in the form of an inverted cone, having a height of 8 m and a base radius of 2 m. Water is flowing into the tank at the rate of $0.5\ \text{m}^3\text{s}^{-1}$. How fast is the water surface area changing when the water in the tank is 2 m deep?

Solution

Let h cm and r cm represent the water level and radius of the water surface respectively.



By similar triangles, $\dfrac{h}{r} = \dfrac{8}{2} \Rightarrow h = 4r$

Volume of water in the cone, $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi r^2(4r) = \dfrac{4\pi}{3}r^3$

$$\Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

Area of water surface, $A = \pi r^2$

$$\Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

We want to find: $\left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{h=2}$

We know: $\dfrac{\mathrm{d}V}{\mathrm{d}r}, \dfrac{\mathrm{d}A}{\mathrm{d}r}, \dfrac{\mathrm{d}V}{\mathrm{d}t}$

So,
$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t} \\
&= \left(\frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}V}\right) \times \frac{\mathrm{d}V}{\mathrm{d}t} \\
&= \left(2\pi r \times \frac{1}{4\pi r^2}\right)(0.5) \\
&= \frac{1}{4r}
\end{aligned}$$

When $h = 2$, $r = 0.5$, $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{1}{4(0.5)} = 0.5$

Hence the water surface area is increasing at a rate of $0.5\ \text{m}^2\text{s}^{-1}$.

Appendix: Further Applications of Differentiation (For Your Information)

1. Velocity and Acceleration

Consider an object moving in a straight line. Let s be the displacement of the object at a time t from a fixed point O. Then

$v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$ is the **instantaneous velocity** of the object at time t;

$a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ is the **instantaneous acceleration** of the object at time t.

Note:

1. Speed is the magnitude of velocity.
2. By convention,

displacement: $\begin{cases} x > 0 & \text{: the object is on the right of } O \\ x < 0 & \text{: the object is on the left of } O \\ x = 0 & \text{: the object is at } O \end{cases}$

velocity: $\begin{cases} v > 0\text{: } x \text{ increases as } t \text{ increases, i.e. the object is moving to the right} \\ v < 0\text{: } x \text{ decreases as } t \text{ increases, i.e. the object is moving to the left} \\ v = 0\text{: the object is at rest} \end{cases}$

acceleration: $\begin{cases} a > 0\text{: } v \text{ increases as } t \text{ increases} \\ a < 0\text{: } v \text{ decreases as } t \text{ increases} \\ a = 0\text{: velocity is constant} \end{cases}$

1.1 Uniformly accelerated motion in a straight line

Since $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$, by integrating, $v = at + C_1$.

If the particle has velocity u when $t = 0$, then $C_1 = u$, giving $v = u + at$.

Since $\dfrac{\mathrm{d}s}{\mathrm{d}t} = v = u + at$, by integrating again

$s = ut + \dfrac{1}{2}at^2 + C_2$ where $C_2 = 0$ since $s = 0$ when $t = 0$.

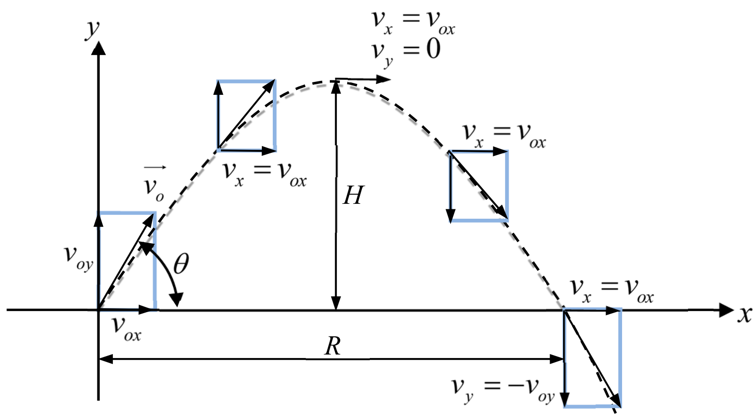
1.2 Motion in a straight line with variable acceleration

Since $\dfrac{\mathrm{d}s}{\mathrm{d}t} = v$, then $s = \int v \, \mathrm{d}t$.

The distance travelled in the interval between $t = t_1$ and $t = t_2$ is given by $\int_{t_1}^{t_2} v \, \mathrm{d}t$. It can also be obtained graphically by evaluating the area between the velocity-time graph and the t-axis between $t = t_1$ and $t = t_2$.

Similarly the change in velocity in the interval between $t = t_1$ and $t = t_2$ is given by $\int_{t_1}^{t_2} a \, \mathrm{d}t$. It can also be obtained graphically by evaluating the area between the acceleration-time graph and the t-axis between $t = t_1$ and $t = t_2$.

2 Projectile Motion



Horizontal distance, $x = v_x t$	Vertical distance, $y = v_{oy} t - \frac{1}{2} g t^2$
Horizontal Velocity, $v_x = v_{ox}$	Vertical Velocity, $v_y = v_{oy} - gt$

Time of flight, $t = \dfrac{2v_0 \sin\theta}{g}$

Maximum height reached, $H = \dfrac{{v_0}^2 \sin^2\theta}{2g}$

Horizontal Range, $R = \dfrac{{v_0}^2 \sin 2\theta}{g}$