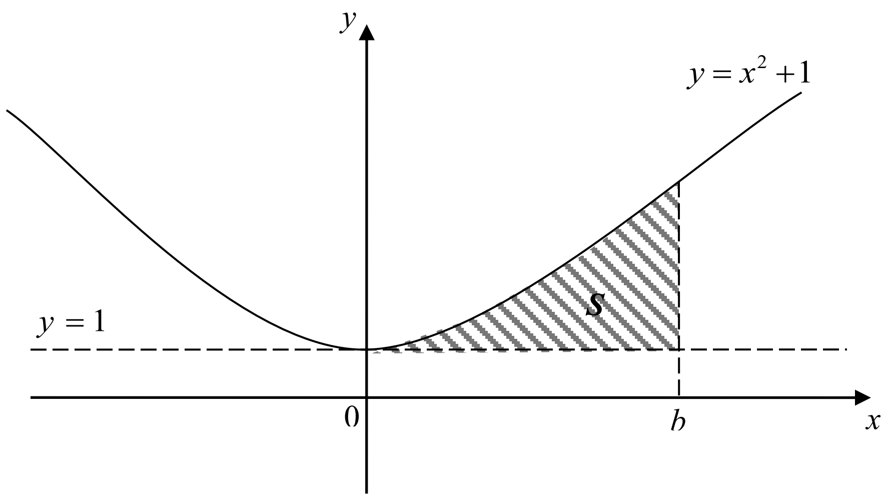
Applications of Integration

1. **CJC/2009/II/2**

(a) Calculate the area of the region bounded by the curves $y = e^{(x-4)} - 1$ and $y = \ln(x) - 1$. [3]

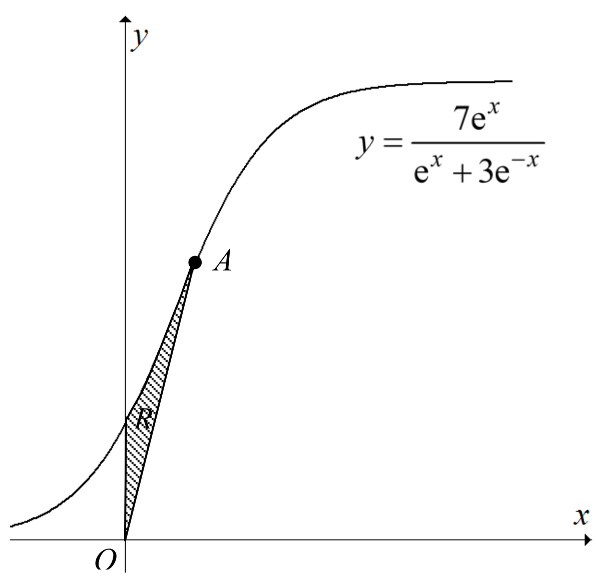
(b) The region S is bounded by the curve $y = x^2 + 1$ and the lines $y = 1$ and $x = b$, where $b > 0$.



V_x is the volume of the solid of revolution formed when S is rotated completely about $y = 1$. V_y is the volume of the solid of revolution when S is rotated completely about the y-axis. Find the value of b such that $V_x = V_y$. [6]

2. **IJC/II/3**

The diagram shows the region R bounded by the curve $y = \dfrac{7e^x}{e^x + 3e^{-x}}$, the y-axis, and the line OA, where A is the point on the curve when $x = \ln 2$.



(i) By means of the substitution $u = \mathrm{e}^x$, find the exact value of $\displaystyle\int_0^{\ln 2} \frac{\mathrm{e}^x}{\mathrm{e}^x + 3\mathrm{e}^{-x}}\,\mathrm{d}x$. [4]

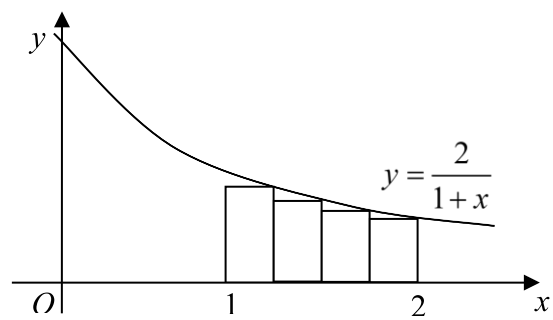
(ii) Calculate the exact area of the region *R*. [2]

(iii) Find the numerical value of the volume of the solid formed when *R* is rotated through 4 right angles about the *x*-axis. [3]

(iv) Find the equation of the curve obtained when the curve $y = \dfrac{7\mathrm{e}^x}{\mathrm{e}^x + 3\mathrm{e}^{-x}}$ is translated 5 units in the negative direction of the *y*-axis, followed by a stretch parallel to the *x*-axis with a scale factor of 3. [2]

3. **PJC/I/9**

The diagram shows the parts of the graph of $y = \dfrac{2}{1+x}$ between $x = 1$ and $x = 2$. Prove that the total area of the four rectangles, each of equal width, as shown in the diagram below, may be expressed as $\displaystyle\sum_{r=1}^{4} \frac{2}{8+r}$. [2]



In general, when there are *n* rectangles, each of equal width, under the curve between $x = 1$ and $x = 2$ (instead of just 4), find an expression, in the form $\displaystyle\sum_{r=1}^{n} \mathrm{f}(r)$, for their total area.

Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2}{2n+r} < 2\ln\left(\frac{3}{2}\right)$.

Deduce that $\displaystyle\sum_{r=1}^{n} \frac{2}{2n+r} < 2\ln\left(\frac{3}{2}\right) < \sum_{r=1}^{n} \frac{2}{2n+r-1}$.

4. **ACJC/2010/I/11**

The region *R* in the first quadrant is bounded by the curve $2y^2 = a(2a - x)$, where $a > 0$, and the line joining $(2a, 0)$ and $(0, a)$. The region *S*, lying in the first quadrant, is bounded by the curve $2y^2 = a(2a - x)$ and the lines $x = 2a$ and $y = a$.

(i) Draw a sketch showing the regions *R* and *S*. [1]

(ii) Find, in terms of *a*, the volume of the solid formed when *S* is rotated completely about the *x*-axis. [4]

(iii) By using a suitable translation, find, in terms of *a*, the volume of the solid formed when *R* is rotated completely about the line $x = 2a$. [4]

5. **AJC/2011/I/4**

(i) By using the substitution $t = \tan x$, show that $\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + c$. [3]

(ii) Find the exact volume of revolution when the region bounded by the curve $y = 2 + \sqrt{\frac{1}{1+\sin^2 x}}$, the lines $x = \frac{\pi}{4}$, $y = 2$ and the y-axis is rotated completely about the line $y = 2$. [3]

6. **IJC/2011/I/11**

(i) Verify, without the use of a calculator, that $\theta = \frac{\pi}{4}$ is a root of the equation

$$\tan^2\theta - 2\cos^2\theta = 0.$$ [1]

(ii) By sketching the graphs of $y = \tan^2\left(\frac{x}{2}\right)$ and $y = 2\cos^2\left(\frac{x}{2}\right)$, solve exactly the inequality

$$\tan^2\left(\frac{x}{2}\right) > 2\cos^2\left(\frac{x}{2}\right)$$

for $-\pi < x < \pi$. [2]

(iii) Hence evaluate exactly $\int_0^{\frac{2\pi}{3}} \left|\tan^2\left(\frac{x}{2}\right) - 2\cos^2\left(\frac{x}{2}\right)\right| \mathrm{d}x$. [6]

7. **PJC/2011/II/5**

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{x}{2}\right)\right)$, giving your answer in its simplest form. [3]

(ii) Hence, or otherwise, show that $\frac{1}{2}\int_0^k \sqrt{4-x^2}\,\mathrm{d}x = a\sqrt{4-k^2} + \sin^{-1}\left(\frac{k}{2}\right)$ where a is to be found in terms of k, $0 < k < 2$. [2]

(iii) Sketch the curve C with equation $4y^2 + x^2 = 4$ and show by shading, a region whose area is given by the integral in **(ii)**. [2]

(iv) Hence, find the area of the region lying inside C and between the lines $x = -1$ and $x = 1$, giving your answer in exact form. [3]

8. **CJC/2013/I/10**

(a) Use the substitution $x = 2\sec\theta$ to find $\int \frac{1}{x^2\sqrt{x^2-4}}\,\mathrm{d}x$.

(b) The region bounded by the curve $y = \frac{1}{\sqrt{1+2x^2}}$, the x-axis, the lines $x = -\sqrt{\frac{3}{2}}$ and $x = -\frac{1}{\sqrt{2}}$ is rotated completely about the x-axis to form a solid of revolution of volume V. Find the exact value of V, giving your answer in the form $k\pi^2$.

9. **MJC/2013/I/9**

(a) Find the exact value of p such that

$$\int_0^{\frac{1}{p}} \frac{1}{1+p^2x^2}\,\mathrm{d}x = \int_1^{\mathrm{e}^2} \ln x\,\mathrm{d}x. \qquad [5]$$

(b) The curve C has equation $y = \dfrac{\sqrt{x}}{1+x^2}$.

(i) Sketch the curve C. [1]

(ii) Use the substitution $u = x^2$ to find $\displaystyle\int_0^n \frac{x}{\left(1+x^2\right)^2}\,\mathrm{d}x$, for $n > 0$. [3]

(iii) Hence find the exact volume of revolution formed when the region between the curve and the positive x-axis is rotated completely about the x-axis. [2]

10. **HCI/2014/I/11**

A curve has equation given by $y = \left(\ln x\right)^2 - 1$, where $x > 0$.

(i) Sketch the graph, indicating the exact coordinates of the x-intercepts and the turning point. [4]

The region R is bounded by the curve and the x-axis.

(ii) Find the exact area of R. [4]

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

11. **TPJC/2014/II/2**

A curve C has equation $y = \dfrac{-x^2+4x+12}{x+3}$.

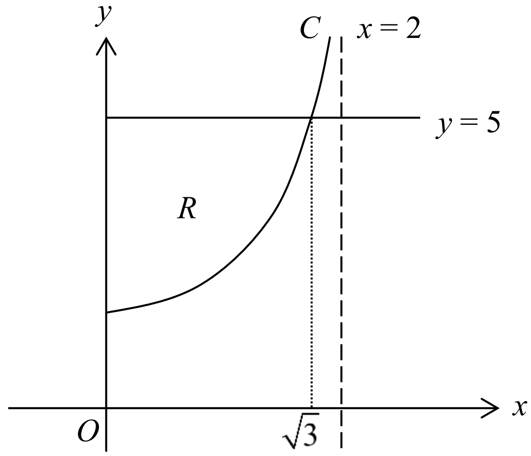
(i) Using an algebraic method, find the range of values of y for which C does not exist. [3]

(ii) Sketch C, stating the exact coordinates of any points of intersection with the axes, the exact coordinates of turning points and the equations of any asymptotes. [3]

(iii) The region enclosed by C, the lines $x = -2$ and $y = 4$ is denoted by R. Find the numerical value of the volume of revolution when region R is rotated through 2π radians about the x-axis. [3]

12. **JJC/2014/I/4**

The diagram shows a region R in the first quadrant bounded by the curve C with equation $y=\dfrac{2}{\sqrt{4-x^2}}+3$, the y-axis and the line y = 5. The line y = 5 and the curve intersect at the point $\left(\sqrt{3},5\right)$.



(i) Calculate the area of region R. [2]

(ii) Write down the equation of the curve obtained when C is translated by 5 units in the negative y-direction. [1]

(iii) Hence show that the volume of the solid formed when R is rotated completely about the line $y=5$ is given by $4\pi\int_0^{\sqrt{3}}\left(\dfrac{1}{4-x^2}-\dfrac{2}{\sqrt{4-x^2}}+1\right)\mathrm{d}x$, and evaluate this integral exactly. [4]

13. **MJC/2014/I/9**

It is given that $\mathrm{f}\left(x\right)=2-\dfrac{1}{x-1}$.

(i) On separate diagrams, sketch the graphs of $y=\mathrm{f}\left(\left|x\right|\right)$ and $y=\left|\mathrm{f}\left(x\right)\right|$, giving the coordinates of any points where the graphs meet the x- and y- axes. You should label the graphs clearly. [4]

(ii) Use a non-calculator method to find the area bounded by $y=\mathrm{f}\left(\left|x\right|\right)$, $x=\dfrac{5}{4}$, $x=2$ and the x-axis. [4]

(iii) The region R is bounded by $y=\left|\mathrm{f}\left(x\right)\right|$, $y=2$ and $x=\dfrac{3}{2}$. Find the volume of revolution formed when R is rotated completely about the y-axis. [4]

14. **RJC/2014/I/7**

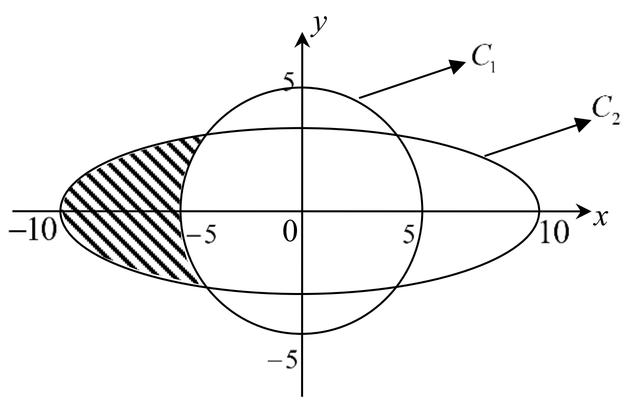
Curves C_1 and C_2 are given by the equations

$$C_1: \quad x^2 + y^2 = 25$$
$$C_2: \quad ax^2 + by^2 = 100$$

where a and b are positive real constants such that $a < b$.

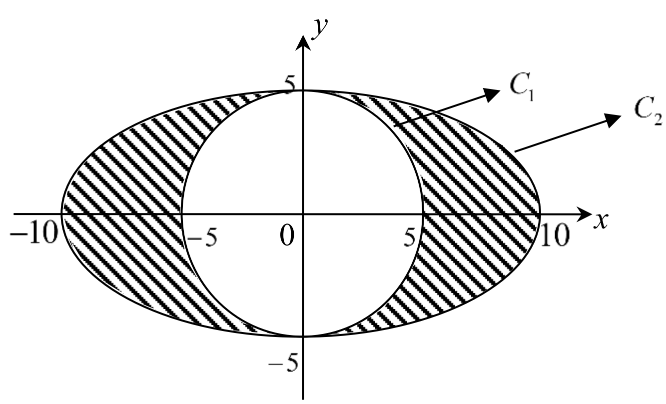
(a) Find the condition(s) on the values of a and b such that C_1 and C_2 intersect at exactly 4 points. [2]

(b) The diagram shows the curves C_1 and C_2 when $a = 1$ and $b = 9$.



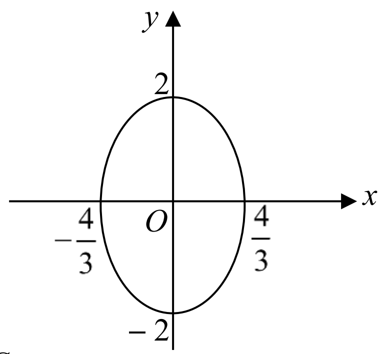
Find the area of the shaded region. [3]

(c) The diagram shows the curves C_1 and C_2 when $a = 1$ and $b = 4$.



Find the exact volume when the shaded region is rotated through π radians about the $y-$axis. [4]

15. **IJC2011/P1/Q5**



The diagram shows the ellipse C_1.

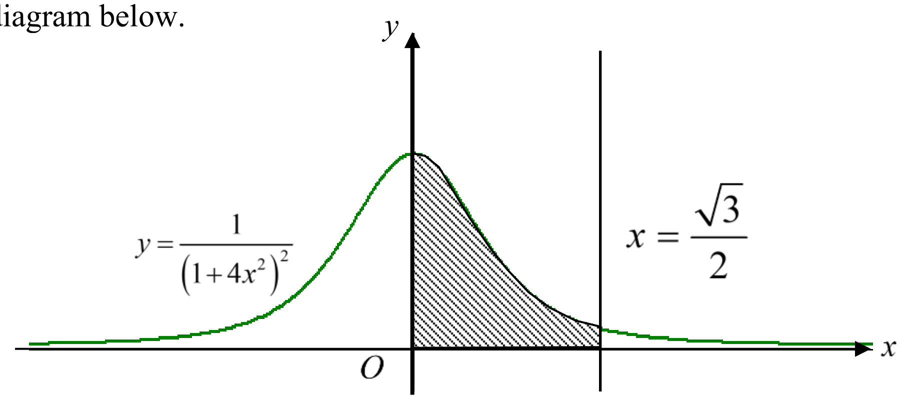
(i) Write down the equation of the ellipse. [1]

(ii) The curve C_2 has equation $y = \dfrac{3x}{4 - x^2}$. Copy the above diagram and sketch the curve C_2 on the same diagram, giving the equations of any asymptotes and the coordinates of the points of intersection of C_1 and C_2. [2]

(iii) The region R in the first quadrant is bounded by the y-axis and the curves C_1 and C_2. Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the x-axis, giving your answer correct to 3 decimal places. [3]

16. **ACJC2011/P1/Q8**

The region R is bounded by the curve $y = \dfrac{1}{\left(1+4x^2\right)^2}$, the line $x = \dfrac{\sqrt{3}}{2}$, the x-axis and the y-axis, as shown in the diagram below.



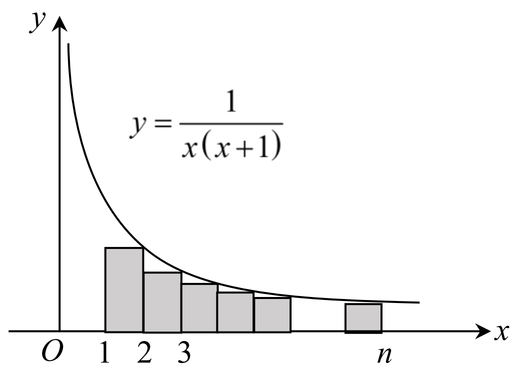
(i) Using the substitution $x = \dfrac{1}{2}\tan t$, find the exact area of R. [6]

(ii) Find the volume of the solid formed when R is rotated completely about the y-axis. [3]

17. **ACJC 2018/P1/Q7**

(i) Express $\dfrac{1}{x(x+1)}$ in partial fractions. [1]

(ii)



The graph of $y=\dfrac{1}{x(x+1)}$, for $0 \le x \le n$, is shown in the diagram. Rectangles, each of width 1 unit, are drawn below the curve from $x = 1$ to $x = n$, where $n \ge 3$.

By considering $\displaystyle\sum_{x=a}^{b} \frac{1}{x(x+1)}$ where a and b are constants to be found, find the total area of the $n - 1$ rectangles in terms of n. [3]

Find the actual area bounded by the curve $y=\dfrac{1}{x(x+1)}$, the x-axis and the lines $x=1$ and $x=n$. [2]

Hence show that $\dfrac{1}{2}-\ln 2<\dfrac{1}{n+1}+\ln\left(1-\dfrac{1}{n+1}\right)$ for all $n \ge 3$. [1]

Using a standard series from the List of Formulae (MF27), show that, for all $n \ge 3$,

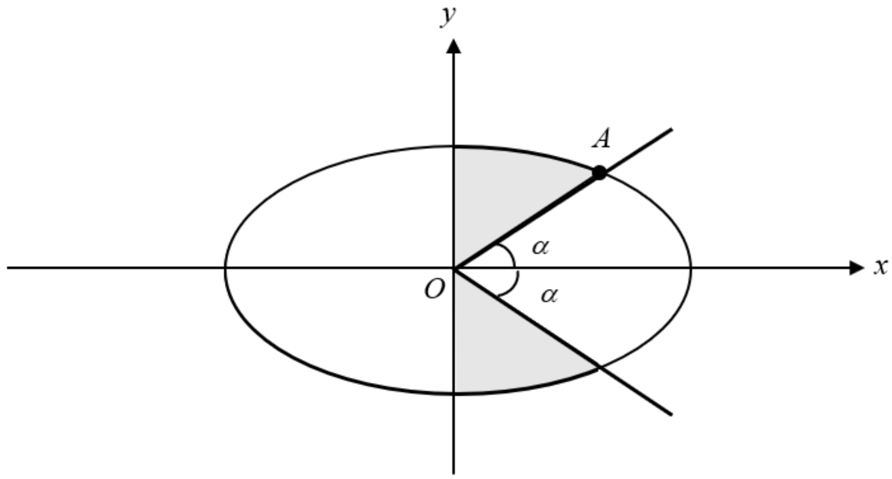
$$\frac{1}{2}-\ln 2<\sum_{r=2}^{\infty}\frac{-1}{r(n+1)^r}.$$ [2]

18. **DHS Prelim/2020/01/Q10**

(i) Use the substitution $x = 3\sin\theta$ to show that

$$\int_0^{x_0} \sqrt{9-x^2}\ \mathrm{d}x = \frac{x_0}{2}\sqrt{9-(x_0)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x_0}{3}\right), \text{ where } 0 \le x_0 \le 3. \qquad [4]$$

The shaded region in the diagram below is enclosed by the curve $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$ for $x \ge 0$, the y-axis and the two lines starting from the origin O, each making an angle α with the x-axis.



Given that $\tan\alpha = \dfrac{2}{3}$ and A is the point of intersection between the line in the first quadrant and the curve, show that the coordinates of A is $\left(\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$.

(ii) Show that the area of the shaded region is $k\pi$, where k is a constant to be determined. [3]

A toy is made by rotating the shaded region completely about the y-axis.

(iii) Find the volume of the toy. [3]

(iv) The toy is now fully enclosed into a cylindrical shape container. State the smallest possible dimensions of the container, assuming that it is made of a material of negligible thickness. [1]

19. **ACJC/2022/I/Q4**

(a) The continuous function $\mathrm{f}(x)$, where $\mathrm{f}(x) > 0$, is strictly decreasing for $x \geq 1$. Sketch the curve $y = \mathrm{f}(x)$ for $k < x < k+1$, where k is an integer and $k \geq 1$.

By comparing the areas of appropriate rectangles and the area under the curve $y = \mathrm{f}(x)$, show that for any integer $k \geq 1$,

$$\mathrm{f}(k+1) < \int_k^{k+1} \mathrm{f}(x)\,\mathrm{d}x < \mathrm{f}(k).$$ [2]

(b) The region under the curve $y = \dfrac{1}{x}$ between $x = 1$ and $x = 10$, is split into 9 vertical strips of equal width. Use the result in part **(a)** to prove

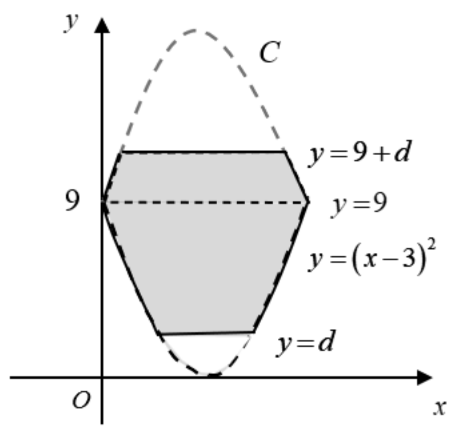
(i) $\displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x < \sum_{k=1}^{9} \frac{1}{k}$, [1]

(ii) $\displaystyle\sum_{k=1}^{9} \frac{1}{k} < 1 + \int_1^{9} \frac{1}{x}\,\mathrm{d}x$. [2]

Hence show that $\ln 10 < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{9} < 1 + \ln 9$. [1]

20. **DHS Prelim 9758/2024/01/Q10(b)**

A chef plans to create an ornament for his master dish. The ornament is made by rotating the shaded region as shown in the diagram completely about the y-axis. The region is bounded by the parabola $y = (x-3)^2$, the curve C and the lines $y = 9 + d$ and $y = d$, where $0 < d < 9$. The curve C is the reflection of the parabola along the line $y = 9$.



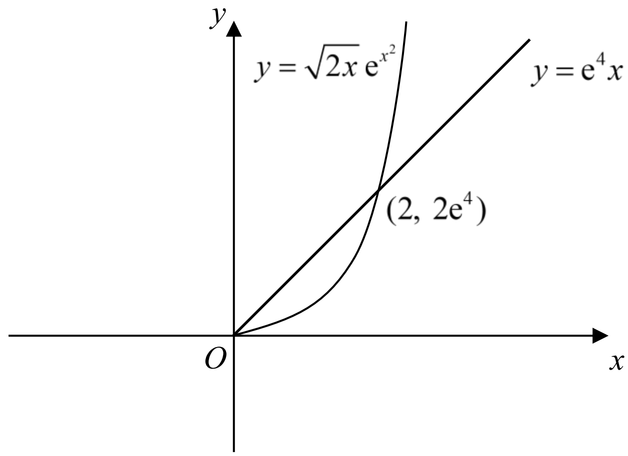
(i) Find the equation of the curve C. [2]

(ii) Find the volume of the ornament in terms of d. [5]

(iii) By varying the value of d, find the maximum volume for the ornament correct to the nearest integer. State the value of d corresponding to this maximum volume. [2]

21. **MI Prelim 9758/2024/01/Q3**

A line l has equation $y = \mathrm{e}^4 x$, $x \geq 0$ and curve C has equation $y = \sqrt{2x}\,\mathrm{e}^{x^2}$, $x \geq 0$. Both l and C intersect at the points with coordinates $(0,\ 0)$ and $(2,\ 2\mathrm{e}^4)$ as shown in the diagram below.



(a) The region R is bounded by the line l, curve C, and the lines $x = 1$ and $x = 3$. Find, correct to 4 significant figures, the area of region R. [3]

(b) The region S is bounded by the line l and curve C. Show that the volume V of the solid formed when S is rotated 2π radians about the x-axis is $V = \frac{\pi}{6}\left(A\mathrm{e}^8 + B\right)$, where A and B are exact constants to be determined. [4]

22. **TJC Prelim 9758/2024/02/Q1**

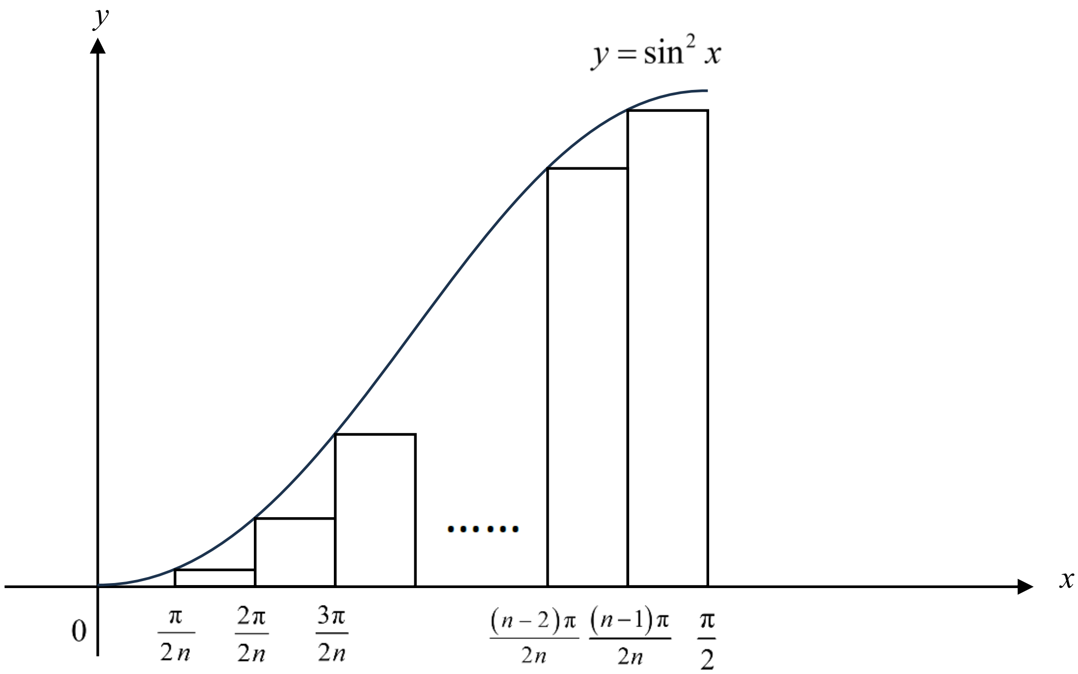
A curve C has equation $y = \dfrac{3-x}{x-1}$.

(a) Sketch C, stating the equations of the asymptotes. [2]

(b) Find the exact volume of the solid generated when the region bounded by C, the x-axis and the line $x = 9$ is rotated through 2π radians about the x-axis. Leave your answer in the form $\pi\left(a + b\ln 2\right)$, where a and b are constants to be determined. [5]

23. **YIJC Prelim 9758/2024/01/Q2**

The diagram below shows a sketch of the graph of $y = \sin^2 x$ for $0 \le x \le \frac{\pi}{2}$. Rectangles each of width $\frac{\pi}{2n}$ are drawn under the curve for $0 \le x \le \frac{\pi}{2}$.



(a) Show that A, the total area of all the rectangles, is given by $a\sum_{k=1}^{n-1} \sin^2 \frac{k\pi}{2n}$, where a is to be determined. [2]

(b) Find the exact value of $\lim_{n\to\infty} a\sum_{k=1}^{n-1} \sin^2 \frac{k\pi}{2n}$. [2]

(c) Hence, find the value of $\int_0^1 \sin^{-1}\sqrt{y}\, \mathrm{d}y$. [2]

24. **YIJC Prelim 9758/2024/02/Q3**

(a) The region R is bounded by the curve $y = \frac{1}{1+\sqrt{x}}$, the x-axis and the lines $x=1$ and $x=4$. R is rotated about the x-axis through 2π radians.
Using the substitution $u=\sqrt{x}$, find the exact volume of the solid generated. [6]

(b) The region L is bounded by the curve $y = \frac{1}{1+\sqrt{x+a}}$, the x-axis and the lines $x=3$ and $x=6$.
Given that the volume of the solid generated when L is rotated about the x-axis through 2π radians is the same as the volume calculated in part **(a)**, state the value of a. [1]

25. **CJC Prelim 9758/2025/P1/Q4**

It is given that

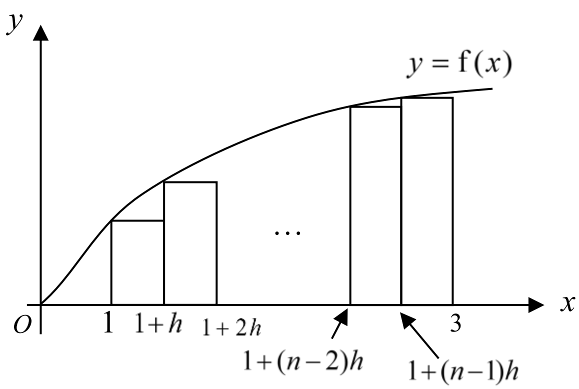
$$\mathrm{f}(x)=\begin{cases} 2-\dfrac{2}{\pi}x & \text{for } 0\le x<\pi \\ x\sin x & \text{for } \pi\le x<2\pi \end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+2\pi)$ for all real values of x.

(a) Sketch the graph of $y=\mathrm{f}(x)$ for $-\pi\le x<4\pi$. You do not need to label the coordinates of any stationary point(s). [3]

(b) Find the exact area bounded by the curve $y=\mathrm{f}(x)$, the x-axis and the lines $x=0$ and $x=\dfrac{3\pi}{2}$. [4]

26. **DHS Prelim 9758/2025/P1/Q10**



The diagram above shows a sketch of the graph of $y=\mathrm{f}(x)$ for $x\ge 0$. There are n rectangles each of width h drawn under the curve for $1\le x\le 3$. Each rectangle when rotated through 2π radians about the x-axis, will result in a cylindrical disc. The total volume of the n cylindrical discs V_1, can be used to estimate the volume V, which is the actual volume generated when the region bounded by the curve, the lines $x=1$ and $x=3$, and the x-axis is rotated through 2π radians about the x-axis.

(a) Show that V_1, the total volume of the n cylindrical discs, is given by $V_1=\pi h\sum_{r=0}^{n-1}\left[\mathrm{f}(1+rh)\right]^2$.

State the value of h in terms of n. [3]

(b) From the above diagram, it can be observed that $V_1<V$. Write down V_2, a similar expression as V_1 where $V_2>V$. [1]

It is now given that $\mathrm{f}(x)=\dfrac{x^2}{\sqrt{1+\mathrm{e}^x}}$ for $x\ge 0$.

(c) Find the value of $\lim_{n\to\infty}V_1$. [2]

(d) Find the area of the region bounded by $y=\mathrm{f}(x)$ and another curve with equation $(x-1)^2+(y-3)^2=9$, for $y\le 3$. [4]

27. **NYJC Prelim 9758/2025/P2/Q3**

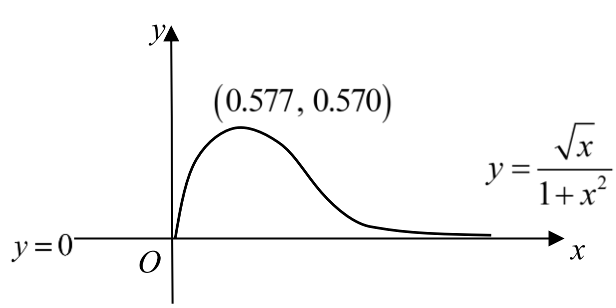
(a) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

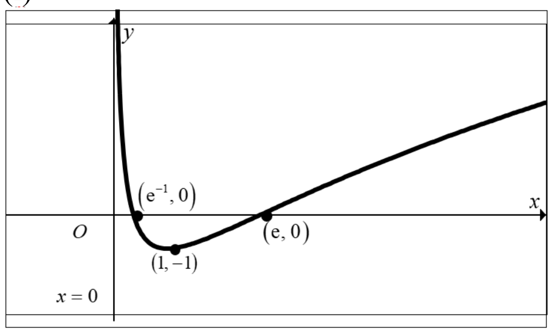
$$\lim_{n\to\infty}\frac{2}{n}\left\{\mathrm{f}\left(1+\frac{2}{n}\right)+\mathrm{f}\left(1+\frac{4}{n}\right)+\ldots+\mathrm{f}\left(1+\frac{2n}{n}\right)\right\}$$

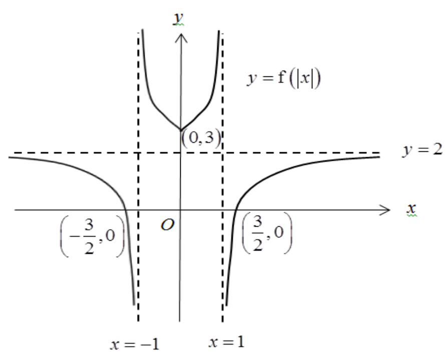
is $\int_1^3 \mathrm{f}(x)\ \mathrm{d}x$. [3]

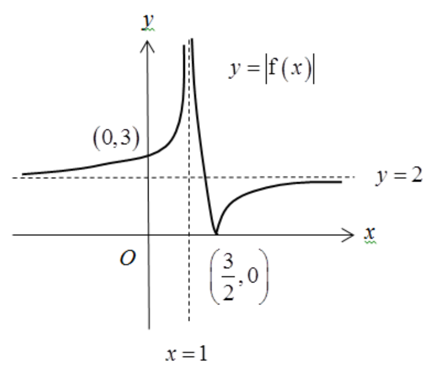
(b) Hence evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{2k}{n}\right)\frac{2}{n}$ exactly. [4]

Answers

1	(i) 1.68 (ii) $b=\frac{5}{2}$
2	(i) $\frac{1}{2}\ln(\frac{7}{4})$ (ii) $\frac{7}{2}\ln(\frac{7}{4})-2\ln 2$ (iii) 6.72 (iv) $y=\frac{7\mathrm{e}^{\frac{1}{3}x}}{\mathrm{e}^{\frac{1}{3}x}+3\mathrm{e}^{-\frac{1}{3}x}}-5$
3	$\sum_{r=1}^{n}\frac{2}{2n+r}$
4	(ii) πa^3 units^3 (iii) $\frac{8}{15}\pi a^3$ units^3
5	(i) $\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\tan x+c$; (ii) $\frac{\pi}{\sqrt{2}}\tan^{-1}\sqrt{2}$.
6	(ii) $-\pi<x<-\frac{\pi}{2}$ or $\frac{\pi}{2}<x<\pi$ (iii) $\frac{3\sqrt{3}}{2}+\frac{2\pi}{3}-2$
7	(i) $2\sqrt{4-x^2}$ (ii) $a=\frac{k}{4}$ (iii) $R=\frac{1}{2}\int_0^k\sqrt{4-x^2}\mathrm{d}x$ (iv) $\sqrt{3}+\frac{2\pi}{3}$
8	(a) $\frac{\sqrt{x^2-4}}{4x}+C$; (b) $\frac{\sqrt{2}}{24}\pi^2$
9	(a) $p=\frac{\pi}{4\left(\mathrm{e}^2+1\right)}$ (b) (i)  (ii) $\frac{1}{2}\left(1-\frac{1}{1+n^2}\right)$ (iii) $\frac{1}{2}\pi$ units^3

<table>
<tr><td>10</td><td>(i)


(ii) $4e^{-1}$; (iii) 12.2</td></tr>
<tr><td>11</td><td>$C = 0$; Tangents to C are vertical lines; $\left(1+\frac{3\pi}{2},1\right)$; $\frac{10}{3}-\frac{3\pi}{4}$;</td></tr>
<tr><td>12</td><td>(i) 1.37 (ii) $y=\frac{2}{\sqrt{4-x^2}}-2$ (iii) $4\pi\left[\frac{1}{4}\ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right| - \frac{2\pi}{3} + \sqrt{3}\right]$</td></tr>
<tr><td>13</td><td>(i)




(ii) Required area $=\frac{1}{2}$
(iii) Required vol $= 2.71$</td></tr>
<tr><td>14</td><td>(a) $b>4$ and $0<a<4$ (b) 22.3; 500π</td></tr>
<tr><td>15</td><td>(i) $9x^2+4y^2=16$ (iii) 9.487</td></tr>
<tr><td>16</td><td>(i) $\frac{\pi}{12}+\frac{\sqrt{3}}{16}$ units^2 (ii) $\frac{3\pi}{16}=0.589$ units^3</td></tr>
<tr><td>17</td><td>(i) $\frac{1}{x}-\frac{1}{x+1}$, (ii) $a=2$, $b=n$, $\frac{1}{2}-\frac{1}{n+1}$, $\ln n-\ln(n+1)+\ln 2$</td></tr>
<tr><td>18</td><td>(ii) $k=\frac{3}{2}$ (iii) 22.1 unit^3 (iv) radius $\frac{3}{\sqrt{2}}$ units and height 4 units</td></tr>
<tr><td>19</td><td>(ii) $\int_1^{10}\frac{1}{x}\,dx<\sum_{k=1}^{9}\frac{1}{k}<1+\int_1^{9}\frac{1}{x}\,dx$</td></tr>
<tr><td>20</td><td>(i) $y=18-(x-3)^2$ (ii) $8\pi\left[54-(d)^{3/2}-(9-d)^{3/2}\right]$ (iii) 877 units^3 when $d=4.5$</td></tr>
</table>

21	(a) 3306 (4 s.f.) (b) $\frac{\pi}{6}\left(13e^{8}+3\right)$
22	(b) $\pi\left(\frac{15}{2}-8\ln 2\right)$
23	(b) $\frac{\pi}{4}$ (c) $\frac{\pi}{4}$
24	(a) $\left(2\ln\frac{3}{2}-\frac{1}{3}\right)\pi$ (b) $a=-2$
25	(b) $(2\pi+1)$
26	(a) $\frac{2}{n}$ (b) $\pi h\sum_{r=1}^{n}\left[f(1+rh)\right]^{2}$ (c) 12.3 (d) 3.01
27	(b) $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{2k}{n}\right)\frac{2}{n}=3\ln 3-2$