Title	Junior College GCE ‘A’ Level H2 Pure Mathematics Material Compilation Series
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Date	12/1/2026

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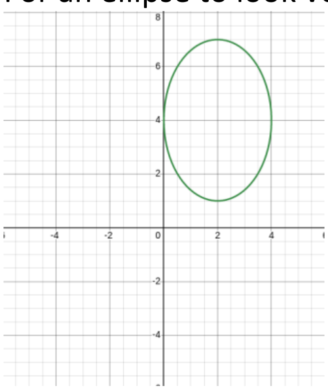
Title	Graphs and Functions – Conic Sections – Hyperbola and Ellipse
Author	AprilDolphin
Date	21/12/2024
Note	This article assume you already understand the properties of circle and parabola as taught in secondary school Additional Mathematics and Elementary Mathematics respectively.

In general, all conic sections can be represented using the equation below,
$Ax^2 + Bx + Cy^2 + Dy + E = 0$
Different types of conic sections appears when we begin changing the value of A, B, C, D and E

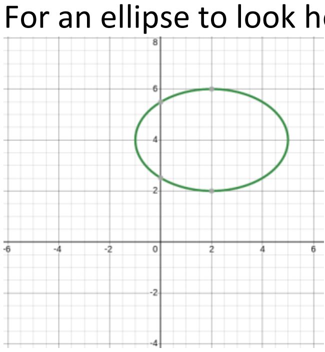
If the conic section is an Ellipse, it has the following equation required in general: (Make sure the right-hand side is exactly 1 before proceeding!)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

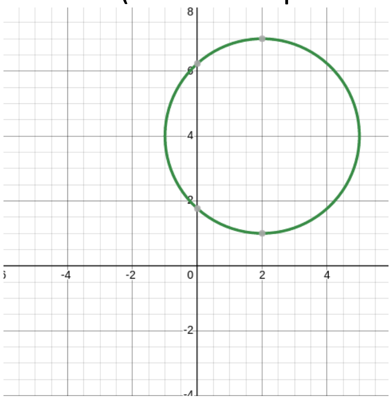
For an ellipse to look vertically stretched, $a < b$



For an ellipse to look horizontally stretched, $a > b$



For an ellipse to have an exactly circular shape, $a = b$, where a represents the radius of a circle. (Circle is a special case of an ellipse.)



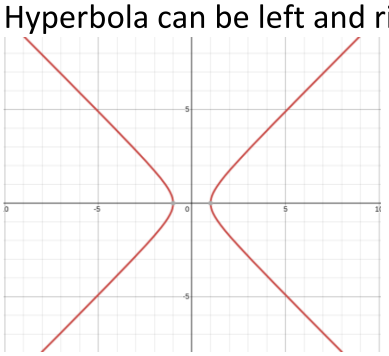
Properties of an ellipse.
The point where the centre lies is $C(h, k)$
The coordinates of vertices from up, down, left and right can be represented as the following

Upper Vertex (Top of Ellipse)	$U(h, k + b)$
Lower Vertex (Bottom of Ellipse)	$D(h, k - b)$
Left Vertex (Left side of Ellipse)	$L(h - a, k)$
Right Vertex (Right side of Ellipse)	$R(h + a, k)$

The lines of symmetry of an ellipse are represented by $x = h$, $y = k$

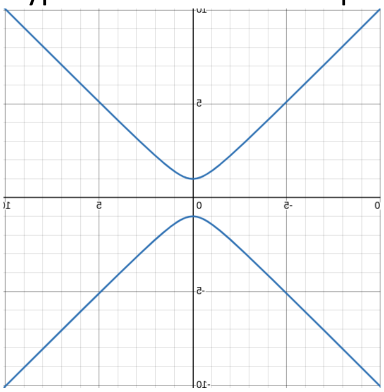
If the conic section is a Hyperbola, it has the following equation in general: (Make sure the right-hand side is exactly 1 before proceeding!)

Hyperbola can be left and right opening as seen with the below diagram



Properties of left-and-right-opening hyperbola as follows:
Can be reduced to the following equation, $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where RHS is exactly 1

Hyperbola can also be up-and-down-opening as seen with the below diagram



Properties of up-and-down-opening hyperbola as follows:

Can be reduced to the following equation, $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$, where RHS is exactly 1

Further Properties of Hyperbola		
	Left-and-Right-Opening	Up-and-Down-Opening
Centre	$C(h, k)$	$C(h, k)$
Vertices	Left Vertex, $L(h-a, k)$ Right Vertex, $R(h+a, k)$	Upper Vertex, $U(h, k+b)$ Bottom Vertex,$B(h, k-b)$
Line of symmetry	$x = h$ $y = k$	$x = h$ $y = k$
Asymptotes	$y = k + \frac{b}{a}(x-h)$ $y = k - \frac{b}{a}(x-h)$	$y = k + \frac{b}{a}(x-h)$ $y = k - \frac{b}{a}(x-h)$

Title	Inequalities Involving Quadratic, Cubic and Rational Functions
Author	AprilDolphin
Date	3/2/2025

Flashback from Secondary School Elementary Mathematics

Description of Rules	**Demonstration**
Addition & Subtraction Rule: The act of adding any numbers to both sides of any inequalities doesn't affect the sign.	$x > 8$ $x + 4 > 8 + 4$ $x \leq 9$ $x + 1 \leq 9 + 1$
Multiplication & Division of Positive Numbers Rule: The act of multiplying or dividing of positive numbers to both sides of any inequalities doesn't affect the sign.	$5x + 8 \geq 9$ $\frac{5x}{5} \geq \frac{1}{5}$ $x \geq \frac{1}{5}$
Multiplication & Division of Negative Numbers Rule: The act of multiplying and dividing of negative numbers to both sides of any inequalities will flip the inequality sign (i.e. $\leq$ changes to $\geq$ and $\geq$ changes to $\leq$)	$2x > 6$ After dividing both sides by -2 , we get $-x < -3$ $\frac{x}{3} \geq 5$ After multiplying both sides by -3, we get $-x \leq -5$

Common Mistakes Made by Inexperienced Students when Solving Inequalities at H2 Level Math.	
Mistake Number 1.	Cross Multiplication $\frac{x-6}{2x+3} > 1$ $x-6 > 2x+3$ Issue: It is incorrect to cross multiply this way as it is currently not known if the sign of $2x+3$ is positive or negative.
Mistake Number 2.	Square Rooting Both Sides of Inequality $x^2 \geq 9$ $x \geq \pm 3$ Issue: It is not correct if you treat the inequality sign as if it were an equal sign for the purpose of square rooting both sides in this way.

Correct Steps to Deal with Inequalities at H2 Mathematics as follows (The steps are to be followed with discretion, as not all steps are required for all question types.)

1. Moving all the terms of inequalities from RHS to LHS, making sure the RHS of the inequality is 0 before proceeding to step 2.
2. Add up everything on LHS (If LHS now has multiple algebraic fractions, add them up into a state where the LHS shares a common denominator.)
3. Factorize the LHS [If LHS cannot be factored, apply quadratic formula to any quadratic portions of the inequality.]
4. If the quadratic portion has no real roots (such as x^2+x+1), complete the square as such portion will be either always positive or always negative.
5. For each linear factor, find the critical value that satisfy the inequality
6. Mark all critical values on the number line, n critical values divide the number line into $n+1$ interval
7. Do a sign test on each interval to determine whether the function is positive or negative in that interval.
8. Complete the sign diagram
9. Determine the solution set, noting that any values that will cause the denominator to be zero <u>must</u> be excluded as <u>division by zero is undefined</u>.

Q1:

Solve the inequality $x^3 \geq x(3x + 10)$.

Step 1. Move terms from RHS to LHS of the inequality.

$x^3 - x(3x + 10) \geq 0$

Step 2. Add up everything on LHS of inequality

$x^3 - 3x^2 - 10x \geq 0$

Step 3. Take out x by factoring.

$x(x^2 - 3x - 10) \geq 0$

Step 4. Further factorize to get the following.

$x(x + 2)(x - 5) \geq 0$

Note: to get sign test result on the bottom right of this page, substitute x on LHS with sign test value.

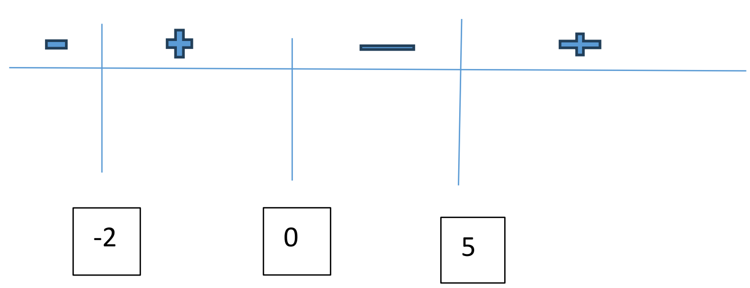
Example if your sign test interval is less than -2, find a value that is less than -2, in this case, I use -3 and substitute x with -3 to get

$(-3)[(-3) + 2][-3 - 5] = -24$

Since the inequality in step 4 indicates the inequality has to be ≥ 0, we look out for set of values that will produce positive or zero. Which get us the following as a result.

$-2 \leq x \leq 0$ OR $x \geq 5$

Sign Test	Result
-3	-24 (Negative)
-1	6 (Positive)
1	-12 Negative
6	48 (Positive)



Q2: Solve the inequality $x^3 < 4x^2 - x$

Step 1. Move terms from RHS to LHS

$x^3 - 4x^2 + x < 0$

Step 2. Factorize the LHS of the inequality

$x(x^2 - 4x + 1) < 0$

Step 3. Use quadratic formula (which I will not further elaborate here) to deal with quadratic portion (Since it cannot be factorized).

$x^2 - 4x + 1 = 0$

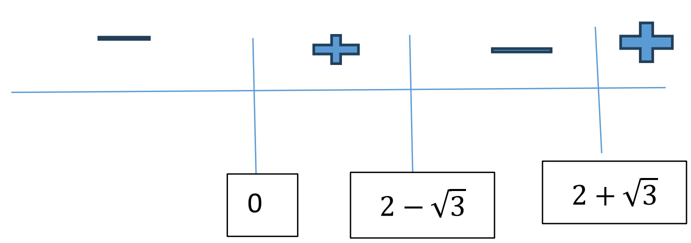
$x = 2 \pm \sqrt{3}$

Final Inequality:

$\left[x - (2 + \sqrt{3})\right]\left[x - (2 - \sqrt{3})\right](x) < 0$

$x < 0$ OR $2 - \sqrt{3} < x < 2 + \sqrt{3}$

Sign Test	Result
-2	-26 (Negative)
0.1	0.061 (Positive
3	-6 (Negative)
4	4 (Positive)



Q3. Solve the inequality $\frac{3x-5}{x-1} \geq \frac{2}{x}$

Step 1: Move all terms from RHS to LHS of inequality

$$\frac{3x-5}{x-1} - \frac{2}{x} \geq 0$$

Step 2: Add up everything on the LHS of inequality

$$\frac{x(3x-5)}{x(x-1)} - \frac{2(x-1)}{x(x-1)} \geq 0$$

$$\frac{3x^2-5x-2x+2}{x(x-1)} \geq 0$$

$$\frac{3x^2-7x+2}{x(x-1)} \geq 0$$

Step 3. Factorize any quadratic portion of the LHS of inequality.

$$\frac{(3x-1)(x-2)}{x(x-1)} \geq 0$$

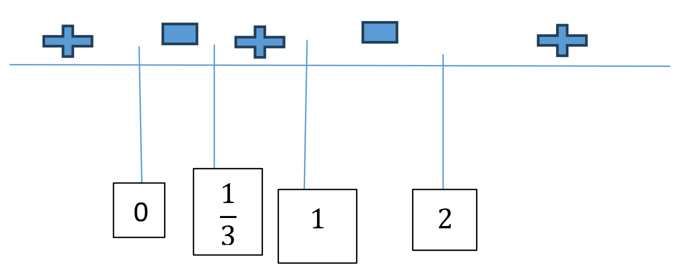
Step 4. Complete sign diagram and exclude any value that will cause the denominator to be 0.

Sign Test	Result
-1	6 (Positive)
0.25	-2.333 (Negative)
0.5	3 (Positive)
1.5	-2.333 (Negative)
3	1.333 (Positive)

$x \neq 0 \;\; AND \; x \neq 1$

Therefore:

$x < 0, \frac{1}{3} \leq x < 1 \; OR \; x \geq 2$



Q4. Solve the following inequality

$\frac{x+1}{x-1} \leq 4$

Step 1. Move terms from RHS to LHS

$$\frac{x+1}{x-1} - 4 \leq 0$$

Step 2. Add up all LHS terms together in the following way

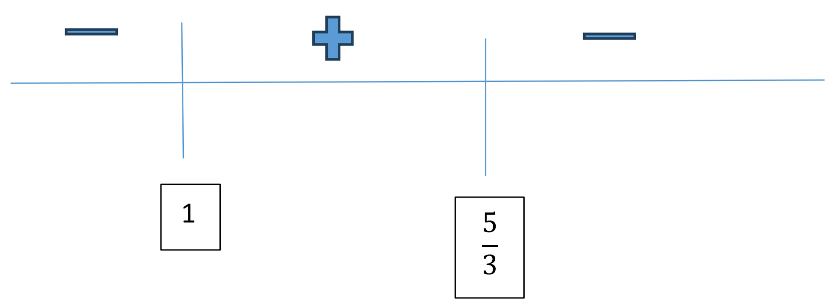
$\frac{x+1}{x-1} - \frac{4(x-1)}{x-1} \leq 0$

$\frac{x+1-4x+4}{x-1} \leq 0$

$$\frac{-3x+5}{x-1} \leq 0$$

Step 3 Complete sign diagram and exclude values that will cause the denominator to be 0.

$x \neq 1$



Therefore

$x < 1 \; OR \; x \geq \frac{5}{3}$

Sign Test	Result
0	-5 (Negative)
1.5	1 (Positive)
2	-1 (Negative)

Title	Inequalities Involving Modulus Function
Author	AprilDolphin
Date	14/2/2025

Basic Concepts of Modulus Functions
The absolute symbol of modulus functions will cause any negative values within the symbol to become positive and any positive values within the absolute symbol remain positive as illustrated in the following examples:
$\lvert\pi\rvert = \pi$
$\lvert -6\rvert = 6$
$\lvert -e^3\rvert = e^3$
$\lvert 15\rvert = 15$
Basic Properties of Equation Involving Modulus Functions
If $\lvert f(x)\rvert = k$, it will also imply $f(x) = \pm k$
Basic Properties of Inequalities Involving Modulus Functions
If $\lvert f(x)\rvert < k$, it will also imply $-k < f(x) < k$
If $\lvert f(x)\rvert > k$, it will also imply $f(x) < -k \ OR \ f(x) > k$

Basic Properties of Operations Involving Modulus Functions
Given a, b are real numbers, the following properties shall apply
$\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$
$\left\lvert\frac{a}{b}\right\rvert = \frac{\lvert a\rvert}{\lvert b\rvert}$
$\lvert a\rvert \pm \lvert b\rvert \neq \lvert a \pm b\rvert$
$\lvert a\rvert^2 = a^2$
If $\lvert a\rvert < \lvert b\rvert, \lvert a\rvert^2 < \lvert b\rvert^2,\ a^2 < b^2$
If $\lvert a\rvert > \lvert b\rvert,\ \lvert a\rvert^2 > \lvert b\rvert^2,\ a^2 > b^2$

Q1. Solve the following inequalities

(a) $|2x+1| > 7$
(b) $|1-x| < 3$
(c) $|2-x| \leq |3+2x|$

1(a)
$|2x+1| > 7$

Using Modulus Function Property:
If $|f(x)| > k$, it will also imply $f(x) < -k \; OR \; f(x) > k$

$2x+1 < -7 \; OR \; 2x+1 > 7$
$2x < -8 \; OR \; 2x > 6$
$x < -4 \; OR \; x > 3$

1(b)
$|1-x| < 3$

Using Modulus Function Property: If $|f(x)| < k$, it will also imply $-k < f(x) < k$

$-3 < 1-x < 3$
$-4 < -x < 2$

Multiplying negative sign to the entire inequality, we get the following:
$4 > x > -2$

1(c)
$|2-x| \leq |3+2x|$

Using Modulus Function Property: If $|a| < |b|$, $|a|^2 < |b|^2$, $a^2 < b^2$

$|2-x|^2 \leq |3+2x|^2$
$[2^2 - 2(2)(x) + x^2] \leq [3^2 + 2(3)(2)x + 4x^2]$
$4 - 4x + x^2 \leq 9 + 12x + 4x^2$
$-5 - 16x - 3x^2 \leq 0$
$-[(3x+1)(x+5)] \leq 0$

Multiplying negative sign to the entire inequality, we get the following:
$(3x+1)(x+5) \geq 0$
$x \geq -\frac{1}{3} \; OR \; x \leq -5$

Q2. Solve the following inequalities

(a) $|x-2| \geq x^2$

(b) $\left|\frac{2x+1}{x-3}\right| > 1$

2(a) $|x-2| \geq x^2$

Using Modulus Function Property:

If $|f(x)| > k$, it will also imply $f(x) < -k \ OR \ f(x) > k$

$x-2 \leq -x^2 \ OR \ x-2 \leq x^2$

$x-2+x^2 \leq 0 \ OR \ x-2-x^2 \leq 0$

$x^2+x-2 \leq 0 \ OR - x^2+x-2 \leq 0$

$(x+2)(x-1) \leq 0$

$-2 \leq x \leq 1$

Multiplying negative sign to the whole inequality, we get

$-x^2+x-2 \leq 0$

$x^2-x+2 \geq 0$

Using quadratic discriminant rule, we calculate the discriminant to get the following: $b^2-4ac = (-1)^2-4(1)(2) = 1-8 = -7 \therefore$ proving that this inequality has no real values that can satisfy itself.

As a result, the only range of value that can satisfy the initial inequality is:

$-2 \leq x \leq 1$

2(b) $\left|\frac{2x+1}{x-3}\right| > 1$

Using Modulus Function Property, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$$\frac{|2x+1|}{|x-3|} > 1$$

Since the denominator is always positive as denoted by the absolute sign, we can be absolutely sure the denominator will not cause any signs to change when multiplied to both sides, we get the following as a result:

$|2x+1| > |x-3|$

Using Modulus Function Property: If $|a| > |b|,\ |a|^2 > |b|^2,\ a^2 > b^2$

$|2x+1|^2 > |x-3|^2 = (2x+1)^2 > (x-3)^2$

$4x^2+4x+1 > x^2-6x+9$

$3x^2 + 10x - 8 > 0$
$x < -4$ OR $x > \frac{2}{3}$

Going back to the original inequality, we have to exclude anything can will cause denominator values to be 0, $\therefore x \neq 3$ and hence there have to be two more solutions that would satisfy the initial inequality, $x < 3$ and $x > 3$

Therefore, $x < -4$, $\frac{2}{3} < x < 3$, $x > 3$

Q3. Solve the following inequality

(a) $\frac{4}{|x|} > x + 3$

Since the numerator of the LHS is positive, we can argue that $|4| = 4$ and put the absolute sign on numerator to get the following value:

$\frac{|4|}{|x|} > x + 3$ which can be rewritten as $\left|\frac{4}{x}\right| > x + 3$

Using Modulus Function Property:
If $|f(x)| > k$, it will also imply $f(x) < -k\ OR\ f(x) > k$

$$\frac{4}{x} < -(x+3)\ \ OR\ \frac{4}{x} > x + 3$$

$$\frac{4}{x} < -x - 3\ OR\ \frac{4}{x} > x + 3$$

$$\frac{4}{x} < \frac{(-x-3)(x)}{x}\ OR\ \frac{4}{x} > \frac{(x+3)(x)}{x}$$

$$\frac{4}{x} < \frac{-x^2 - 3x}{x}\ \ OR\ \frac{4}{x} > \frac{x^2 + 3x}{x}$$

$$\frac{4}{x} - \frac{-x^2 - 3x}{x} < 0\ \ OR\ \frac{4}{x} - \frac{x^2 + 3x}{x} > 0$$

$$\frac{4 + x^2 + 3x}{x} < 0\ OR\ \frac{4 - x^2 - 3x}{x} > 0$$

$x^2 + 3x + 4$ has a discriminant value of $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 = -7$ and $a > 0$. Therefore, no values could satisfy the inequality $4 + x^2 + 3x < 0$

$\frac{1}{x} < 0$

$x < 0$

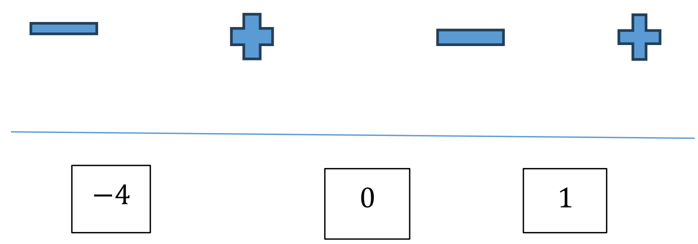
$$\frac{-x^2 - 3x + 4}{x} > 0$$

Multiplying both sides by -1, we get the following

$$\frac{x^2 + 3x - 4}{x} < 0$$

Factorizing the numerator, we get the following

$$\frac{[(x + 4)(x - 1)]}{x} < 0$$



$x < -4$ OR $0 < x < 1$

Sign Test	Result
-5	-1.2 (Negative)
-2	3 (Positive)
0.5	-4.5 (Negative)
2	3 (Positive)

$\therefore x < 0$ or $0 < x < 1$

Q4. $\frac{|x|+5}{2-|x|} < 3$

Substitute $u = |x|$ and we get the following:

$$\frac{u+5}{2-u} < 3$$

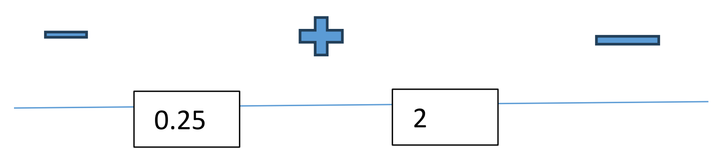
$$\frac{u+5}{2-u} - 3 < 0$$

$$\frac{u+5}{2-u} - \frac{3(2-u)}{2-u} < 0$$

$$\frac{u+5-6+3u}{2-u} < 0$$

$$\frac{4u-1}{2-u} < 0$$

$u < \frac{1}{4}$ OR $u > 2$



Sign Test	Results
0	-0.5 (Negative)
0.5	0.667 (Positive)
3	-11 Negative

For $u = |x| < \frac{1}{4}$, can be rewritten as $-\frac{1}{4} < x < \frac{1}{4}$

For $u = |x| > 2$, can be rewritten as $x < -2$ OR $x > 2$

$$\therefore x < -2 \ \ OR \ \ -\frac{1}{4} < x < \frac{1}{4} \ OR \ x > 2$$

Title	Differentiation of Inverse Trigonometric Functions
Author	AprilDolphin
Date	19/9/2024
Assumptions	This article assumes you already have good knowledge of using chain rule to differentiate functions as well as basic knowledge of finding derivatives of algebraic, logarithmic, exponential and trigonometric functions.

Function	Derivatives
$f(x) = \sin^{-1}(x)$	$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$
$f(x) = \cos^{-1}(x)$	$f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$
$f(x) = \tan^{-1}(x)$	$f'(x) = \dfrac{1}{1+x^2}$

Chain Rule in General
Given $f(x) = g\big(h(x)\big)$ $f'(x) = g'(x)\,h'(x)$

Application of Chain Rule to Inverse Trigonometric Functions	
Function	Derivatives
$f(x) = \sin^{-1}(u)$	$f'(x) = \dfrac{1}{\sqrt{1-u^2}}(u')$
$f(x) = \cos^{-1}(u)$	$f'(x) = \dfrac{-1}{\sqrt{1-u^2}}(u')$
$f(x) = \tan^{-1}(u)$	$f'(x) = \dfrac{1}{1+u^2}(u')$

Example 1.
Find the derivative of the following functions

(a) $y = \sin^{-1}(e^{6t})$

Using chain rule $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$

$$\frac{dy}{du} = \frac{1}{\sqrt{(1-u^2)}} \times 6e^{6t} = \frac{6e^{6t}}{\sqrt{1-e^{12t}}}$$

(b) $y = \tan^{-1}(2^x)$

Using chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{1}{1+(2^x)^2} \times 2^x \ln(2) = \frac{2^x \ln(2)}{1+2^{2x}}$$

(c) $f(x) = \cos^{-1}\left(\frac{x}{3}\right)$

Using chain rule, where any composite function $f(x) = g\big(h(x)\big)$ will produce a derivative of $g'(x)h'(x)$.

$$\cos^{-1}\left(\frac{x}{3}\right) = \cos^{-1}\left(\frac{1}{3}x\right)$$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1}{3}x\right)^2}} \times \frac{1}{3} = -\frac{1}{3\sqrt{1-\frac{1}{9}x^2}} = -\frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}$$

$$= -\frac{1}{\sqrt{9-\frac{\sqrt{9^2}x^2}{9}}} = -\frac{1}{\sqrt{9-x^2}}$$

(d) $y = e^{\sin^{-1}(2x)}$

Using chain rule, where $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$2e^{\sin^{-1}(2x)} \times \frac{1}{\sqrt{1-(2x^2)}} = \frac{2e^{\sin^{-1}(2x)}}{\sqrt{1-4x^2}}$$

Title	Differentiation of Implicit Functions
Author	AprilDolphin
Date	27/9/2024

Definition of an explicit function, is where
y can be defined as exactly equal to $f(x)$ or written mathematically, $y = f(x)$

Definition of an implicit function is where
y cannot be written as equal to solely just $f(x)$ but requiring the person to write or rewrite a function as where $g(x, y) = 0$

Examples of Explicit Functions	Examples of Implicit Functions
$y = x^2 + \ln(x) + 2x + 4$	$x^2 + y^2 + 2xy = 25$
$y = \sin(x^4) + x^5 - \tan(x)$	$y^3 - \frac{5x}{y} = 6x^2y^6 + 9x - 64$
$y = e^{\sin(x)} - \cos(x) + 5$	$y^5 + \sin(xy) = x^2 + y^5$

Rules of Differentiation Necessary to Ensure a Smooth Understanding of Implicit Differentiation:

Chain Rule: $f(x) = g\big(h(x)\big), f'(x) = g'(x)\ h'(x)$

Product Rule: $f(x) = g(x)\ h(x), f'(x) = g'(x)\ h(x) + h'(x)\ g(x)$

Quotient Rule: $f(x) = \frac{g(x)}{h(x)}, f'(x) = \frac{h(x)g'(x) - g(x)h\prime(x)}{[h(x)]^2}$

Instruction to Implicit Differentiation

1. Differentiate left side and right side of implicit function with respect to x
2. Identify term where y is differentiated, multiplying the derivative of $f(y)$ by $\frac{dy}{dx}$
3. Group terms with $\frac{dy}{dx}$ together on one side and without on the other side
4. Take out common factor of $\frac{dy}{dx}$ on the side with the term of $\frac{dy}{dx}$
5. Remove terms without the factor of $\frac{dy}{dx}$ by division, multiplication or any applicable method

Example 1.
Find the derivative of the following implicit function
$y^2x^2 + x = y - 10$

Step 1. Differentiate left hand side and right-hand side of implicit function with respect to x.

$$\frac{d}{dx}\,[y^2x^2 + x] = \frac{d}{dx}[y - 10]$$

Step 2. Differentiating term by term, we get the following and can identify the term where y is differentiated then multiply the derivative by $\frac{dy}{dx}$

Terms	Derivatives
y^2x^2 using product rule →	$[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2)$
x using power rule →	1
y	$1\times \frac{dy}{dx}$
-10	0

Putting all derivative together, we would get the following value:
$$[2y(x^2)] \times \frac{dy}{dx} + (2x)(y^2) + 1 = 1 \times \frac{dy}{dx}$$

Step 3. Group terms with $\frac{dy}{dx}$ together on one side and without on the other side

$$2yx^2\left(\frac{dy}{dx}\right) + 2xy^2 + 1 = \frac{dy}{dx}$$

$$2yx^2\left(\frac{dy}{dx}\right) - \frac{dy}{dx} = -2xy^2 - 1$$

Step 4. Take out common factor of $\frac{dy}{dx}$ on the side with the term of $\frac{dy}{dx}$

$$(2yx^2 - 1)\left(\frac{dy}{dx}\right) = -2xy^2 - 1$$

Step 5. Remove terms without the factor of $\frac{dy}{dx}$ by division, multiplication or any available method

$$\frac{dy}{dx} = \frac{-2xy^2 - 1}{2yx^2 - 1}$$

Example 2. Differentiate the following implicit function
$y^2 = 2x^3 + y + 7$

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[2x^3 + y + 7]$$

$$2y\left(\frac{dy}{dx}\right) = 6x^2 + 1\left(\frac{dy}{dx}\right) + 0$$

$$2y\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 6x^2$$

$$\frac{dy}{dx}(2y - 1) = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{2y - 1}$$

Example 3.
$2xy + y^2 - 3x - x^2y^5 = 0$

$$\frac{d}{dx}[2xy + y^2 - 3x - x^2y^5] = \frac{d}{dx}[0]$$

Terms	Derivative
$2xy$	$2x(1) \times \frac{dy}{dx} + 2y$
y^2	$2y \times \frac{dy}{dx}$
$-3x$	-3
$-x^2y^5$	$-2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx}$

0	0

$$2x\left(\frac{dy}{dx}\right) + 2y + 2y\left(\frac{dy}{dx}\right) - 3 - 2xy^5 + 5y^4(-x^2) \times \frac{dy}{dx} = 0$$

$$2x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + 2y - 3 - 2xy^5 - 5x^2y^4\left(\frac{dy}{dx}\right) = 0$$

$$2x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) - 5x^2y^4\left(\frac{dy}{dx}\right) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx}(2x + 2y - 5x^2y^4) = -2y + 3 + 2xy^5$$

$$\frac{dy}{dx} = \frac{-2y + 3 + 2xy^5}{2x + 2y - 5x^2y^4}$$

Example 4.

$$y + y^2 + xy = 5 - \sin(e^{3y})$$

$$\frac{d}{dx}[y + y^2 + xy] = \frac{d}{dx}[5 - \sin(e^{3y})]$$

$$1\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + y + x\left(\frac{dy}{dx}\right) = 0 - 3e^{3y}\cos e^{3y}\left(\frac{dy}{dx}\right)$$

$$1\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) + x\left(\frac{dy}{dx}\right) + 3e^{3y}\cos(e^{3y})\left(\frac{dy}{dx}\right) = -y$$

$$\frac{dy}{dx}[1 + 2y + x + 3e^{3y}\cos(e^{3y})] = -y$$

$$\frac{dy}{dx} = -\frac{y}{1 + 2y + x + 3e^{3y}\cos(e^{3y})}$$

Title	Parametric Differentiation
Author	AprilDolphin
Date	11/11/2024

Definition of Parametric Functions

A parametric function is a function that express relationships of 2 variables using a third variable, in which the third variable is called a parameter.

Example of a Parametric Function as follows
$y = t^2 + 5t - 4$
$x = 2t + 6$

In the above case, the relationship between x and y is expressed using parameter t.

Simplest examples of applications of parametric functions:

Kinematics Modeling – where a flying object (such as a bird or fighter jet) trajectory creates a situation where the object's forward motion speeds up and the height follows a curve that goes up and down repeatedly.

In this above case, it would be extremely difficult to express the variables in the non-parametric way and using a parametric function for kinematic modelling is appropriate in this case in the following manner.

The forward motion (being in acceleration) over time can be written as
$x(t) = 0.8t + 3t^2$

The altitude of the flying object in up and down motion over time can be written as
$y(t) = 5 + \cos(t)$

Differentiation involving parametric functions can be done using a technique derived from chain rule.

Let x be the first variable, y be the second variable and t be the parameter involved and our goal is to find the change in y with respect to x.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dt} \times \frac{dt}{dx}$$

1. Find $\frac{dy}{dx}$ given that $y = 4t^2 + 3t - 1$ and $x = 3t^4 - 2t$

$$\frac{dy}{dt} = 8t + 3$$

$$\frac{dx}{dt} - 12t^3 - 2$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{(8t + 3)}{(12t^3 - 2)}$$

2. Find $\frac{dy}{dx}$ for the parametric equation $x = 2\sin(t)$ and $y = 3\cos(t)$

$$\frac{dy}{dt} = -3\sin(t)$$

$$\frac{dx}{dt} = 2\cos(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3\sin(t)}{2\cos(t)} = -\frac{3}{2}\tan(t)$$

3. A curve is defined parametrically by $x = \frac{2t}{t+1}$ and $y = \frac{t^2}{t+1}$.
   Find $\frac{dy}{dx}$.

Using parametric differentiation, we can find the derivatives of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ as follows.

Quotient rule where $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

$$\frac{dy}{dt} = \frac{(t+1)(2t) - 1(t^2)}{(t+1)^2} = \frac{2t^2 + 2t - t^2}{(t+1)^2} = \frac{t^2 + 2t}{(t+1)^2}$$

$$\frac{dx}{dt} = \frac{(t+1)(2) - 2t(1)}{(t+1)^2} = \frac{2t + 2 - 2t}{(t+1)^2} = \frac{2}{(t+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 + 2t}{(t+1)^2} \times \frac{(t+1)^2}{2} = \frac{t^2 + 2t}{2}$$

Title	Integration by Parts
Author	AprilDolphin
Date	26/10/2024

Formula for Integration by Parts

$$\int u\,dv = uv - \int v\ du$$

In any integration by parts application, we choose the value of u using a rule where the type of function mentioned first has the highest priority and the one mentioned second has lower priority and so on, the one mentioned the last will have the lowest priority.

The rule is called
"Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, Exponential"

Highest Priority → Lowest Priority

In short, the rule is called the L.I.A.T.E rule.
The purpose of complying to the above rule is to ensure the function we selected is made easy by integration by parts rather than being made even more complicated. As seen with the below example.

Example 1.

$$\int x\,e^x\ dx$$

If we select $u = x$, in compliance with the rule mentioned above, we literally get the following:

$u = x$ and $dv = e^x$

$$v = \int e^x = e^x$$

$\int v = e^x$

$du = 1\,dx$

As a result, we combined them together using the integration by parts formula

$$\int u\,dv = uv - \int v\ du$$

$$x\,e^x - e^x(1) = xe^x - e^x + C$$

However, if we reversed the roles of u and dv above, we get the following

$u = e^x\ , dv = x$

$du = e^x\ , v = \frac{x^2}{2}$

Combining them together, using the integration by parts formula, we get the following integral, literally,

$$e^x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\ e^x\,dx$$

Which becomes as difficult or even harder to integrate compared to the function we first being with and that's not what we want.

Question 1.
Find $\int \ln x\ \ dx$

Which can be rewritten as

$$\int \ln x\ (1)\ dx$$

Using the L.I.A.T.E rule explained earlier, we can choose $u = \ln x$ and therefore, $dv = 1$.

$u = \ln x$
$v = x$

$$\int x\frac{1}{x}\ dx\ = \int 1\ \ dx = x$$

Combining them, we get $x\ln(x) - x + C$
Question 2.
Find $\int x^2 \ln x\ \ dx$

Selecting $u = \ln x$ and therefore $dv = x^2$

$$v = \int x^2 = \frac{x^3}{3}$$

$$\int v\ du = \int \frac{x^3}{3}\left(\frac{1}{x}\right) = \int \frac{1}{3}x^2 = \frac{\left(\frac{1}{3}\right)}{2+1}x^{2+1} = \frac{1}{9}x^3$$

$$\ln(x)\left(\frac{x^3}{3}\right) - \frac{x^3}{9} + c$$

Question 3.
Find $\int x \cos x\ dx$

Selecting $u = x$ and therefore $dv = \cos x$

$$v = \int \cos x = \sin x$$

$$\int v\, du = \int \sin(x)\ 1\ dx = -\cos x$$

$$x \sin x + \cos x + C$$

Question 4.
Find $\int \tan^{-1} x\ \ dx$

Select $u = \tan^{-1} x$ and $dv = 1\ dx$

$$v = x$$

$$\int v\ du = \int x\left(\frac{1}{x^2+1}\right) = \int \frac{x}{x^2+1} = 0.5\ln(x^2+1)$$

$$x\tan^{-1}(x) - 0.5\ln(x^2+1)$$

Title	Differential Equations – Basic Concepts and First Order Differential Equations (Direct Integration and Separation of Variables)
Author	AprilDolphin
Date	30/10/2024

Differential Equations are equations that contain derivatives, where the so-called "Solutions" are the function(s) that can satisfy the derivatives. The act of finding the indefinite integral of derivatives is generally considered solving simple differential equations.

Concepts to understand in any differential equations

- Order – the order of a differential equation is simply the order of the highest derivative in the differential equation.
- Degree – the degree of a differential equation is simply the power of the highest derivative of the differential equation.

Example

1. Given the following three examples, state the order and degree of the differential equation:

   (a) $x\left(\frac{d^3y}{dx^3}\right) + (3y)^2 = 6x - 8$

   (b) $3x(y'')^2 - y^2y' = 0$

   (c) $x\left(\frac{dy}{dx}\right) + 8\left(\frac{d^2y}{dx^2}\right)^3 = 3x - 5y$

1(a) Highest derivative is the third derivative – third order differential equation
Highest power of the highest derivative is 1 – first degree differential equation
∴ Third order, first degree differential equation.

1(b) Highest derivative is the second derivative – second order differential equation
Highest power of the highest derivative is 2 – second degree differential equation
∴ Second order, second degree differential equation

1(c) Highest derivative is the second derivative – second order differential equation
Highest power of the highest derivative is 3 – third degree differential equation
∴ Second order, third degree differential equation.

After understanding the above, we can go on and understand the concept of independent and dependent variables.

Independent variables in any differential equation are the variables with respect to where differentiation occur and dependent variables are the variables being differentiated.

In the below example, identify the dependent and independent variables.
Example

2.

$$x\left(\frac{d^3y}{dx^3}\right) + (3y)^2 = 6x - 8$$

y is the dependent variable

x is the independent variable

Q1. Solve the following differential equation using direct integration

(a) $\frac{\mathrm{dy}}{\mathrm{dx}} = \ln x$

Move dx to the other side by multiplying both sides by dx to get the following
$dy = \ln x \, dx$

Integrate both sides to get

$$\int dy = \int \ln x \ dx$$

Using integration by parts for the right-hand side, we get the following

Set $u = \ln x$, $dv = 1$
$v = x$

$$du = \frac{1}{x}$$

$$\int v \ du = \int \frac{x}{x} = \int 1 = x$$

$y = x(\ln x\,) - x + C$

Q2. Solve the following differential equation by separation of variables.

$$x + y\left(\frac{dy}{dx}\right) = 2$$

$$y\left(\frac{dy}{dx}\right) = 2 - x$$

Multiply both sides by dx

$$y\,dy = 2 - x\,dx$$

Integrate both sides to get the following

$$\int y\,dy = \int 2 - x\,dx$$

$$\frac{y^2}{2} = 2x - \frac{x^2}{2}$$

$$y^2 = 4x - x^2 + C$$

Q3. Solve the following differential equation by separation of variables.

$$\frac{dy}{dx} = \frac{x^2}{y^3}$$

Cross-multiply both sides denominator and we get the following

$$y^3\,dy = x^2\,dx$$

$$\int y^3\,dy = \int x^2\,dx$$

$$\frac{y^4}{4} = \frac{x^3}{3}$$

Cross multiply again to get

$$3y^4 = 4x^3$$

Divide both sides by 3

$$y^4 = \frac{4}{3}x^3 + C$$

Title	Complex Numbers – Basic Operations and Basic Knowledge
Author	AprilDolphin – Lim Wang Sheng
Date	5/1/2026

Perhaps earlier in your schooling life before entering an 'A' Level course, you may have encountered equations such as $x^2 + 1 = 0$ for which you may be told by your teachers there are no real roots to the equation as mentioned above. The topic of complex numbers is designed so as to fill such a gap in knowledge where certain equation will produce no real solutions under secondary level Mathematics knowledge.

I would like to introduce the concept of imaginary number i, which is defined as exactly equal to $\sqrt{-1}$.
(If you have read through some of my secondary school Mathematics notes, I provided a recommendation to students to avoid using letters like i as much as possible to prevent confusion with the definition of i at 'A' Level Mathematics.)

Warning for 'A' Levels students pursuing an engineering course in future:
For students who are interested in reading engineering books to pursue a course such as an engineering related degree, I would like to warn that some engineering books use letter j to represent the concept of $\sqrt{-1}$ instead mostly to prevent confusion with the letter I which is used to represent current in engineering books. Please do not use j to represent imaginary number under any circumstances at A Levels.

Letter Notation Used	Meaning of Notation
N	Natural Numbers
Z	Integers
Q	Rational Numbers
Q^c	Irrational Numbers
R	Real Numbers
C	Complex Numbers

Notation Used in Complex Number Topic Exclusively	Meaning, Description or Explanation
Re(z), where z is a complex number	Real Part of a Complex Number Given a Complex Number $z = a + bi$, Re$(z) = a$
Im(z), where z is a complex number	Imaginary Part of a Complex Number Given a Complex Number $z = a + bi$, Im$(z) = b$

z^*, where z is a complex number	The conjugate of z, where $z = a + bi$, $z^* = a - bi$

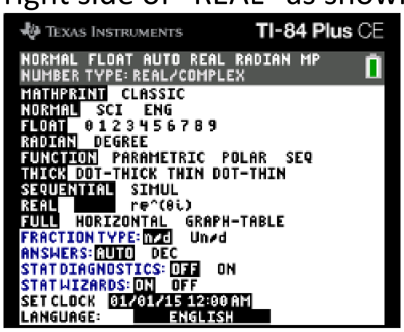
Preparation of TI-84 Plus CE Calculator for the Purpose of Dealing with Complex Numbers.

After turning on your TI-84 Plus CE or TI-84 Plus CE Python Calculator, please follow the steps as outlined in this paragraph.

Step 1. Press [MODE] Button

Step 2. After pressing [MODE] Button, press down arrow key on your Graphing Calculator until the cursor reaches "REAL" as shown in the below exhibit.

Step 3. Press the right arrow key once so that the cursor moves to "$a + bi$" just on the right side of "REAL" as shown in the below exhibit.



Step 4. Once done, press [CLEAR] button to leave the mode setup screen.

(IMPORTANT NOTE: Once the calculator is reset, the process mentioned above will need to be repeated.)

Q1. Evaluate the following

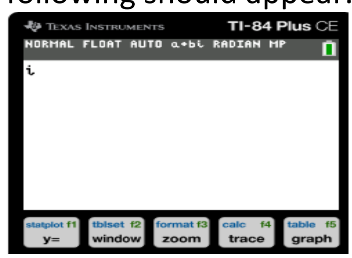
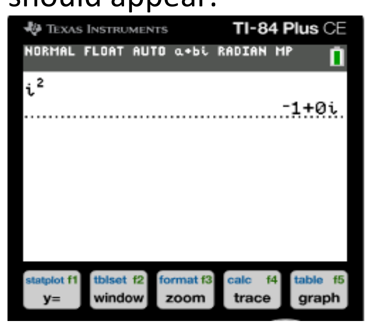
(a) $\sqrt{-64}$

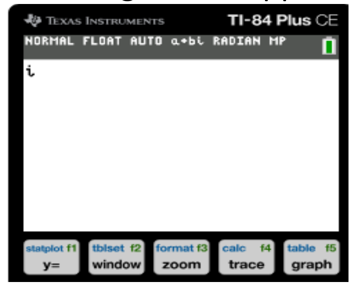
(b) $\sqrt{-2}$

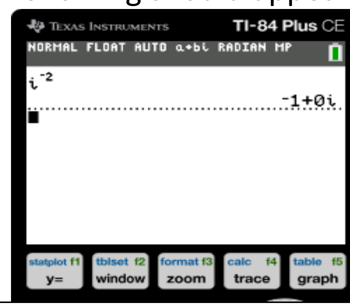
Steps on Paper	Steps on TI-84 Plus CE (if applicable)
1(a)$\sqrt{-64} = \sqrt{64}(\sqrt{-1}) = 8i$	Press [2ND] and Press [x^2] button Press "-64" Press [ENTER] and the answer "$8i$" should appear
1(b)$\sqrt{-2} = \sqrt{2}(\sqrt{-1}) = \sqrt{2}i$	

Q2. Evaluate the following:

(a) i^2
(b) i^{-2}
(c) i^3
(d) i^{-3}
(e) i^4
(f) i^5
(g) i^{-5}

Steps on Paper	Steps on TI-84 Plus CE (if applicable)
2(a) $i^2 = (\sqrt{-1})^2 = -1$	Press [2ND] and Press [.] button and the following should appear.  Press [^] button and followed by [2], and press [ENTER] button and the following should appear.  The answer is $-1 + 0i$ which is basically just -1

<table>
<tr>
<td>2(b) $i^{-2} = \frac{1}{i^2} = \frac{1}{i^2} \times \frac{i^2}{i^2} = \frac{-1}{1} = -1$</td>
<td>Press [2ND] and Press [.] button and the following should appear.


Press [^] button and followed by “-2”, and press [ENTER] button and the following should appear.

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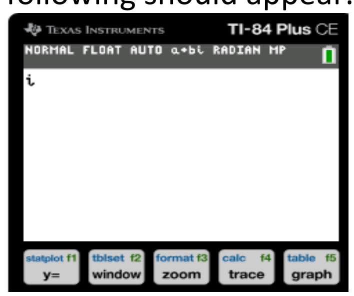
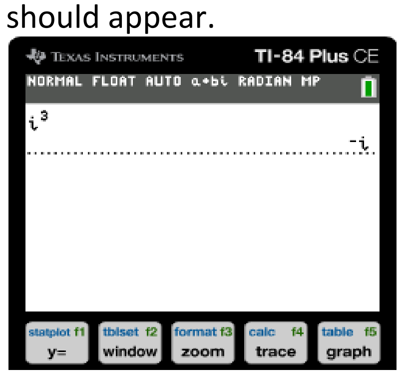
<tr>
<td>2(c) $i^3 = i^2(i) = -i$</td>
<td>Press [2ND] and Press [.] button and the following should appear.


Press [^] button, followed by “3” and press [ENTER] button and the following should appear.

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<td>

2(d)$i^{-3} = \frac{1}{i^3} = \frac{1}{i^2} \times \frac{1}{i} =$

$\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$

$-i \times \frac{1}{i^2} = -i \times \frac{1}{-1} = i$

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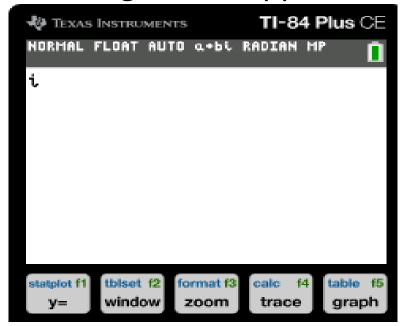
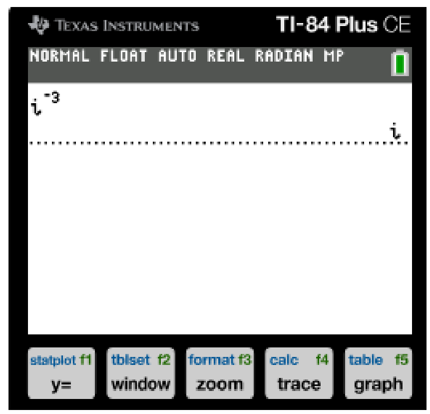
Press [2ND] and Press [.] button and the following should appear.



Press [^] button followed by "-3" and press [Enter] button to get the answer as follows.



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<tr>
<td>

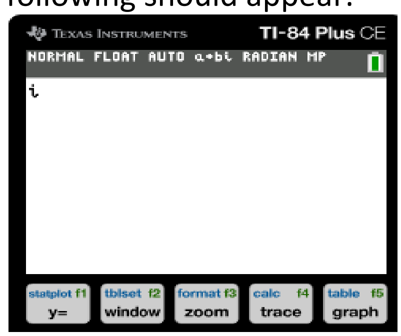
2(e) $i^4 =$

$i^2 \times i^2 =$

$-1(-1) = 1$

</td>
<td>

Press [2ND] and Press [.] button and the following should appear.



Press [^] button followed by "4" and press [Enter] button to get the answer as follows.

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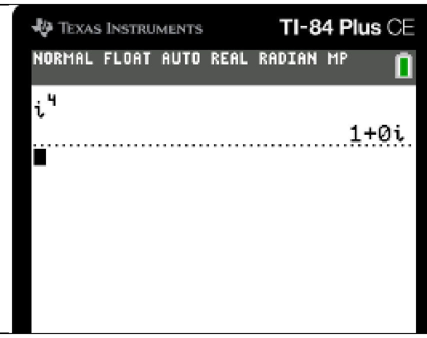
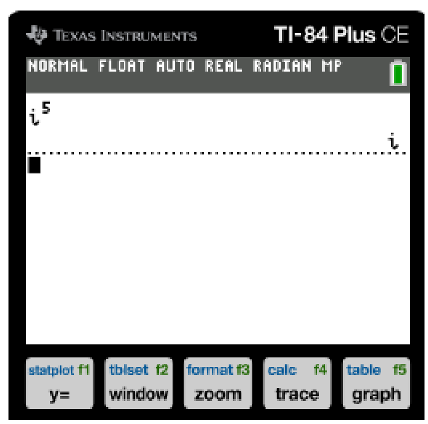
<td>
</td>
</tr>
<tr>
<td>2(f) $i^5 =$

$i^2 \times i^2 \times i = 1 \times i$

$= i$</td>
<td>Press [2ND] and Press [.] button and the following should appear.


Press [^] Button, followed by “5” and press enter, the answer should appear as follows.

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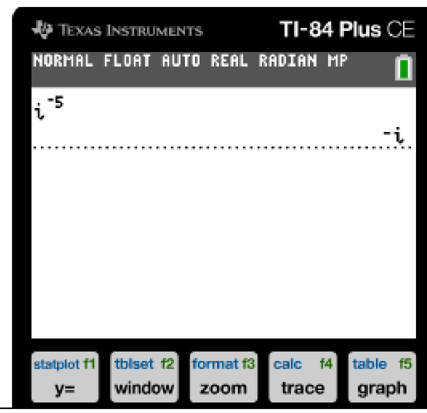
<tr>
<td>2(g)
$i^{-5} =$

$$\frac{1}{i^5} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$</td>
<td>Press [2ND] and Press [.] button and the following should appear.

Press [^] button followed by “-5” and press [Enter] button to obtain the answer as follows.

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</table>

After a while of overview of the concept of imaginary numbers, we have to also look into the concept of complex numbers, complex numbers are numbers with both imaginary number part and real number part, in this topic the complex numbers shall only be given in $a + bi$ form.

Q3(a) Solve the quadratic equation $z^2 + z + 1 = 0$

<table>
<tr>
<td>Method 1: Completing the Squares
$z^2 + z + 1 = 0$
$$z^2 + z + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2 = 0$$
$$\left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$$</td>
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</table>

$\left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$ $\left(z+\frac{1}{2}\right)=\pm\sqrt{-\frac{3}{4}}$ $z=\pm\sqrt{-\frac{3}{4}}-\frac{1}{2}$ $z=-\frac{1}{2}+\sqrt{\frac{3}{4}}i$ OR $z=-\frac{1}{2}-\sqrt{\frac{3}{4}}\ i$	
Method 2: Quadratic Formula Given $z^2+z+1=0$ $z=\frac{-b\pm\sqrt{b^2-4ac}}{2(a)}$ $z=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$ $z=-\frac{1}{2}+\frac{\sqrt{3}}{2}\ i$ OR $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	

Basic Four Operations on Complex Numbers (Addition, Subtraction, Multiplication and Division)

For addition and subtraction on complex numbers it will be similar to how you study how to addition and subtraction of algebraic expression in secondary school where like terms are added together.

Q4(a) Evaluate $(3+6i)+(1-6i)$
(b) Evaluate $(2+i)-(6-5i)$

Q4(a) $(3+6i)+(1-6i)=4+0i=4$
Q4(b) $(2+i)-(6-5i)=-4+6i$

For multiplication involving complex numbers, it will be similar to how you multiply algebraic expressions together but with a few new rules needed to note here:

$i \times i = -1$

$-i \times i = 1$

Q5(a) Evaluate $(1 - 4i)(3 + 5i)$

$$3 + 5i - 12i - (20)(i)(i) = -7i + 3 + 20(1) = 23 - 7i$$

For division involving complex numbers, it will be rather similar to how you divide expression involving surds at Secondary Additional Mathematics, we simply multiply the conjugate of the denominator over the conjugate of the denominator.

Q6(a) Evaluate $\frac{3+i}{1+2i}$ in exact cartesian form.

In the above case, we find the conjugate of the denominator divided by the conjugate of the denominator, which is $\frac{1-2i}{1-2i}$ and multiply the value to the expression given in the question as follows:

$$\frac{3+i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{(3+i)(1-2i)}{(1+2i)(1-2i)} =$$

$$\frac{3 + i - 6i - 2i(i)}{1^2 - (2i)^2} = \frac{5 - 5i}{1 - (4)i^2} = \frac{5 - 5i}{5}$$

$$= 1 - i$$

Q6(b) Evaluate $\frac{2+9i}{5-2i}$ in exact cartesian form.

$$\frac{2+9i}{5-2i} \times \frac{5+2i}{5+2i} = \frac{(2+9i)(5+2i)}{(5-2i)(5+2i)} =$$

$$\frac{10 + 45i + 4i + 18(i)^2}{25 - (2i)^2} = \frac{10 - 18 + 45i + 4i}{25 + 4}$$

$$= \frac{-8 + 49i}{29} = \frac{-8}{29} + \frac{49}{29}i$$

Q7(a) Solve simultaneously for w and z: $2w + z = 12i$

$$w^* + 2z = \frac{4i-13}{2-i}$$

Equation 1: $2w + z = 12i$

Equation 2: $w^* + 2z = \frac{4i-13}{2-i}$

Evaluate the algebraic complex number fraction portion in equation 2

$$\frac{4i-13}{2-i} \times \frac{2+i}{2+i} = \frac{(4i-13)(2+i)}{(2-i)(2+i)} = \frac{-30-5i}{5} = -6-i$$

Equation 2A: $w^* + 2z = -6 - i$

Equation 1A: $2w + z = 12i$ is to be rewritten as $z = 12i - 2w$

Equation 1B: $2z = 24i - 4w$

Substitute 1B into 1A

Equation 2B: $w^* + 24i - 4w = -6 - i$

Rewritten w and w^* in $x + yi$ and $x - yi$ format respectively, we get the following Equation

$(x - yi) + 24i - 4(x + yi) = -6 - i$

$x - yi + 24i - 4x - 4yi = -6 - i$

$x - yi - 4x - 4yi = -6 - 25i$

$-3x - 5yi = -6 - 25i$

$-3x = -6$

$x = 2$

$-5yi = -25i$

$y = 5$

$w = 2 + 5i$

$2z = 24i - 4(2 + 5i) = -8 + 4i$

$z = -4 + 2i$

Q8 Express $(2-i)^3$ in the form $a+bi$, where a, b are real numbers.

Written Steps	Steps on TI-84 Plus CE
$(2-i)^3 =$ $\binom{3}{0}2^0(-i)^{3-0}+\binom{3}{1}2^1(-i)^{3-1}$ $+\binom{3}{2}2^2(-i)^{3-2}+\binom{3}{3}2^3(-i)^{3-3}=$ $i-6-12i+8=2-11i$	Press the open bracket key "(" Press $2-i$ Press the close bracket key ")" Press "^" key and press "3" Press enter and the answer should appear as $2-11i$

Q9(a)

Given that $(x+yi)^2 = i$, determine the possible values of real numbers x and y, giving your answer in exact form.

$(x+yi)^2 = i$
$x^2+2xyi+(yi)^2 = i$
$\therefore x^2-y^2 = 0$ [Equation 1]

$2xyi = i$
$2xy = 1$
$x = \frac{1}{2y}$ [Equation 2]
$x^2 = \left(\frac{1}{2y}\right)^2 = \frac{1}{4y^2}$ [Equation 2B]

Substitute Equation 2B into Equation 1

$$\frac{1}{4y^2} - y^2 = 0$$

Multiply $4y^2$ throughout the equation 2B to get the following
$1-4y^4 = 0$

Factorise and we get
$(1-2y^2)(1+2y^2) = 0$
$\therefore 1 = 2y^2$ or $1 = -2y^2$
$y^2 = \frac{1}{2}$ or $y^2 = -\frac{1}{2}$
$y = \pm\sqrt{\frac{1}{2}}$ or $y = \pm\sqrt{-\frac{1}{2}}$ (Rejected)

$y = \sqrt{\frac{1}{2}}$ or $y = -\sqrt{\frac{1}{2}}$

$x = \frac{1}{2y}$ or $x = \frac{1}{2y}$

$x = \frac{1}{2\sqrt{\frac{1}{2}}}$ or $x = \frac{-1}{2\sqrt{\frac{1}{2}}}$

Two important theorems required for study of complex numbers, namely Fundamental Theorem of Algebra and Conjugate Roots Theorem.

Fundamental Theorem of Algebra
Any polynomial that can be written in the form of $a_0 + a_1z + a_2z^2 + a_3z^3 \dots a_{n-1}z^{n-1} + a_nz^n = 0$ will have exactly n number of roots, taking into consideration repeated and complex roots.

Conjugate Roots Theorem
Given any polynomial equations with all real coefficients, the conjugate roots theorem implies the following: If $w = a + bi$ is a complex solution to the polynomial, the conjugate $w^* = a - bi$ will also be a complex solution to the polynomial

The table below illustrates all possible solutions types when dealing with polynomials with degree greater than 1.

Polynomial Degree	Possible Results of Finding Solutions
2	2 Real Roots 2 Complex Roots (1 Conjugate Pair of Complex Roots)
3	3 Real Roots 1 Real Root + 2 Complex Roots (1 Conjugate Pair of Complex Roots)
4	4 Real Roots 2 Real Roots + 2 Complex Roots (1 Conjugate Pair of Complex Root) 4 Complex Roots (2 Conjugate Pair of Complex Roots)
5	5 Real Roots 3 Real Roots + 2 Complex Roots (1 Conjugate Pair of Complex Roots) 1 Real Roots + 4 Complex Roots (2 Conjugate Pair of Complex Roots)

Q10 (a)

Given that complex number $1 - i$ is a solution to the equation $z^3 - 4z^2 + 6z - 4 = 0$, find the remaining roots of the equation.

By conjugate roots theorem, we can establish the following fact:
If $z = 1 - i$ is a root of the equation, $z^* = 1 + i$ must also be a root as well.
So, we can establish that

$z - (1 - i)$ is a factor and thus $z - (1 + i)$ is also a factor of the equation mentioned
$(z - 1 + i)(z - 1 - i) = z^2 - 2z + 2$

Long Division Method

$$
\begin{array}{r|l}
 & z - 2 \\
\hline
z^2 - 2z + 2 & z^3 - 4z^2 + 6z - 4 \\
 & -(z^3 - 2z^2 + 2z) \\
\hline
 & -2z^2 + 4z - 4 \\
 & -(-2z^2 + 4z - 4) \\
\hline
 & 0
\end{array}
$$

Since the long division approach as seen produces the result where $z - 2$ is a factor of the cubic equation in this question, we can determine that $z = 2$ since $z - 2 = 0$.
Hence the cubic equation $z^3 - 4z^2 + 6z - 4 = 0$ has roots $z = 2$ as well as $z = 1 + i$

Q10 (b)

Given that $-1 - i$ is a root of the equation $z^3 - z^2 - 4z + p = 0$, find the constant p
Using the factor theorem as taught in Secondary School level of Additional Mathematics, where if any factor within a polynomial will produce a result of $P(x) = 0$, we can establish that the following has to be true:
$P(-1 - i) = (-1 - i)^3 - (-1 - i)^2 - 4(-1 - i) + p = 0$
$6 + p = 0$
$p = -6$

Q11 The equation $z^3 + az^2 + bz + c = 0$, where a, b and c are constants has roots 3 and w where w is a complex number.

(a) State a condition on a, b and c for the third root to be w^*.

(b) Given that the condition in part (a) holds, and that $w = -1 + 2i$, find the values of a, b and c.

Q11(a) The values of a, b and c <u>all</u> has to be real numbers.

Q11(b)
By conjugate root theorem, we understand that
If $w = -1 + 2i$ is a root, then the conjugate $w^* = -1 - 2i$ is also a root.

Rewritten the value of w as $z = -1 + 2i$, $z^* = -1 - 2i$ and $z = 3$,
$(z + 1 - 2i)(z + 1 + 2i)(z - 3)$

$(z^2 + 2z + 1 + 4)(z - 3) =$
$(z^2 + 2z + 5)(z - 3) =$
$z^3 + 2z^2 + 5z - 3z^2 - 6z - 15 =$
$z^3 - z^2 + 5z - 6z - 15 =$
$z^3 - z^2 - z - 15$

$a = -1, b = -1, c = -15$

Q12 Given that the quartic equation $z^4 - 3z^3 + 8z^2 - 7z + 5 = 0$ has a complex root $1 - 2i$, solve the equation exactly.

A few things to note in this question

1. By Fundamental Theorem of Algebra, it is guaranteed that a polynomial of degree 4 have exactly 4 roots in total.
2. By Conjugate Roots Theorem, if $w = 1 - 2i$ is a root, $w^* = 1 + 2i$ is also a root as well.

Knowing these 2 facts, we can proceed to find the other 3 roots.
$z = 1 - 2i$ and $z = 1 + 2i$

So, $[z - (1 - 2i)][z - (1 + 2i)] = 0$

$z^2 - (1 - 2i)z - (1 + 2i)z + 5 = 0$

$z^2 - z - 2iz - z + 2iz + 5 = 0$
$z^2 - 2z + 5 = 0$

By Long Division Method

$$
\begin{array}{r}
z^2 - z + 1 \\
z^2 - 2z + 5 \,\big)\, \overline{z^4 - 3z^3 + 8z^2 - 7z + 5} \\
-\left(z^4 - 2z^3 + 5z^2\right) \\
\hline
-z^3 + 3z^2 - 7z \\
-\left(-z^3 + 2z^2 - 5z\right) \\
\hline
z^2 - 2z + 5 \\
-\left(z^2 - 2z + 5\right) \\
\hline
0
\end{array}
$$

After using long division method, we get the other quadratic factor of $z^2 - z + 1$ which gives us the two quadratic factors of $(z^2 - z + 1)(z^2 - 2z + 5) = 0$ from the original question.

$z^2 - z + 1 = 0$ is to be solved using quadratic formula as follows:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$z = \frac{1 \pm \sqrt{-3}}{2}$$

$z = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ or $z = \frac{1}{2} - \frac{\sqrt{-3}}{2}$

$z = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ or $z = \frac{1}{2} - \frac{\sqrt{3}i}{2}$

$\therefore$ The roots are $z = 1 - 2i$, $z = 1 + 2i$, $z = \frac{1}{2} + \frac{\sqrt{3}i}{2}$ and $z = \frac{1}{2} - \frac{\sqrt{3}i}{2}$