HWA CHONG INSTITUTION
2025 JC2 Preliminary Examination

MATHEMATICS
Higher 3

9820/01

Paper 1

Wednesday **22 September 2025** **3 hours**

Additional materials: 12-page Answer Booklet
List of Formula (MF26)
4-page Additional Answer Booklet (upon request)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the 12-page Answer Booklet and any other additional 4-page Answer Booklets you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF26).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, slot any additional 4-page Answer Booklets used in your 12-page Answer Booklet and indicate on the 12-page Answer Booklet the number of additional 4-page Answer Booklets used (if any).

This question paper consists of 6 printed pages and 0 blank page.

1. For $n \in \{2^k,\ k \in \mathbb{Z}^+\}$, prove by mathematical induction that $\dfrac{x_1 + x_2 + \ldots + x_n}{n} \geq (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$,

where $x_1, x_2, \ldots x_n$ are positive real numbers. **[6]**

2. Let S_i be the set containing all permutations of n objects where the i^{th} object is in the i^{th} position, i.e. the i^{th} object is in its original position.

(a) Find the values of $\sum_{i=1}^{n} |S_i|$ and $\sum_{i<j} |S_i \cap S_j|$ in terms of n. **[4]**

A derangement is a permutation of the elements of a set in which no element appears in its original position.

(b) Using the principle of inclusion-exclusion, show that the number of derangements of n objects, $D_n = \left|\overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_n}\right|$ is given by

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$ **[3]**

(c) State the value of $\lim_{n\to\infty} \dfrac{D_n}{n!}$ in exact form. **[1]**

3. Let $\mathrm{f}(n) = 2\left[1 + \dfrac{1}{(1+\mathrm{i})^2}\right]\left[1 + \dfrac{1}{(1+2\mathrm{i})^2}\right]\ldots\left[1 + \dfrac{1}{(1+n\mathrm{i})^2}\right]$, where $\mathrm{i} = \sqrt{-1}$.

(a) By showing $1 + \dfrac{1}{(1+k\mathrm{i})^2} = \dfrac{[1+(k-1)\mathrm{i}][1+(k+1)\mathrm{i}]}{(1+k\mathrm{i})^2}$ or otherwise, find an expression of $\mathrm{f}(n)$, leaving your answer in Cartesian form in terms of n. **[6]**

(b) Using the fact that if $w, z \in \mathbb{C}$, then $\arg(wz) = \arg(w) + \arg(z)$, find the value of n such that $\tan\left[\sum_{k=0}^{n} \arg\left(1 + \dfrac{1}{(1+k\mathrm{i})^2}\right)\right] = -\dfrac{36}{37}$. **[3]**

4. Let w, x, y and z be positive real numbers.

(a) Show that $\dfrac{x^2}{x+y}+\dfrac{y^2}{y+z}+\dfrac{z^2}{z+w}+\dfrac{w^2}{w+x} \geq \dfrac{w+x+y+z}{2}$. **[3]**

(b) Deduce that $\dfrac{x^2}{x+y}+\dfrac{y^2}{y+z}+\dfrac{z^2}{z+w}+\dfrac{w^2}{w+x} \geq 2\left(wxyz\right)^{\frac{1}{4}}$ **[2]**

(c) Suppose $wx = 27$ and $yz = 3$, show that $\dfrac{x^2}{x+y}+\dfrac{y^2}{y+z}+\dfrac{z^2}{z+w}+\dfrac{w^2}{w+x} > 6$. **[3]**

5. Let $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Then the function $\mathrm{f}: \mathbb{R} \to \mathbb{Z}$ where $\mathrm{f}(x) = \lceil x \rceil$ is called the ceiling function. For example, $\mathrm{f}(\pi) = \lceil \pi \rceil = 4$, $\mathrm{f}(-4.5) = \lceil -4.5 \rceil = -4$ and $\mathrm{f}(3) = \lceil 3 \rceil = 3$.

(a) Sketch the graph of $y = \lceil 2\sin(2x) \rceil$, for $0 \leq x \leq 2\pi$. **[4]**

(b) By considering the period, show that

$$\int_0^{2\pi} \lceil 2\sin(nx) \rceil \, \mathrm{d}x = k\pi$$

for all $n \in \mathbb{Z}^+$, where k is an integer to be determined. **[3]**

(c) Hence, deduce the value of $\int_0^{2\pi} \lceil n\sin(nx) \rceil \, \mathrm{d}x$ in terms of π. **[2]**

6. **(a)** The function f, for $x \neq 0$, satisfies the equation

$$2\mathrm{f}(x) - \mathrm{f}\left(\frac{1}{x}\right) = 3x.$$

(i) Explain why

$$2\mathrm{f}\left(\frac{1}{x}\right) - \mathrm{f}(x) = \frac{3}{x}.$$ **[1]**

(ii) Hence find $\mathrm{f}(x)$. **[2]**

(b) The function g, for $x \neq 0$, satisfies the equation

$$\mathrm{g}(x) + \mathrm{g}(-x) - \mathrm{g}\left(\frac{1}{x}\right) = x.$$

Find $\mathrm{g}(x)$. **[6]**

7. **(a)** Factorise $x^5 + 1$.

Prove that if $2^m + 1$ is prime, then m must be of the form 2^k, where $k \in \mathbb{Z}$. **[3]**

(b) Let p be a prime factor of $F_n = 2^{2^n} + 1$.

By considering the smallest positive integer d such that $2^d \equiv 1 \pmod{p}$, or otherwise, prove that p must be of the form $k2^{n+1} + 1$, where k is an integer. [You may use Fermat's Little Theorem: $a^{p-1} \equiv 1 \pmod{p}$, where a and p are coprime] **[3]**

(c) Deduce that any factor of $F_n = 2^{2^n} + 1$ must be of the form $k2^{n+1} + 1$, where $k \in \mathbb{Z}$.

[2]

(d) Using the fact that $641 = 5^4 + 2^4 = 5(2^7) + 1$, show that $5^4\, 2^{28} + 2^{32} \equiv 0 \pmod{641}$ and $5^4\, 2^{28} - 1 \equiv 0 \pmod{641}$.

Deduce that $F_5 = 2^{2^5} + 1$ is divisible by 641. **[3]**

8. Differentiation from First Principles and the q-Derivative: A Comparative Perspective

The concept of the derivative, as we understand it today, was formalised in the 17th century through the independent work of Isaac Newton and Gottfried Wilhelm Leibniz, marking the birth of calculus. Central to this development is the **first principle of differentiation**, which defines the derivative of a function f at a point x as:

$$\mathrm{f}'(x)=\lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$$

This expression captures the idea of the instantaneous rate of change or the slope of the tangent line to the curve at a given point. For example, to find the derivative of $\mathrm{f}(x)=x^n$ using the first principle of differentiation, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} \\ &= \lim_{h\to 0}\frac{x^n+\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\ldots+h^n-x^n}{h} \\ &= \lim_{h\to 0}\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h+\ldots+h^{n-1} \\ &= nx^{n-1} \end{aligned}$$

While powerful, this approach assumes the ability to take infinitesimal increments $h\to 0$, which is inherently tied to the real number continuum. In the 20th century, especially in the study of quantum calculus — calculus without limits — mathematicians explored finite difference analogues. One of the most prominent constructions is the **q-derivative**, introduced by Frank Hilton Jackson in 1908. The **q-derivative** of a function f(x), for a real number $q\neq 1$, is defined as:

$$D_q\,\mathrm{f}(x)=\frac{\mathrm{f}(qx)-\mathrm{f}(x)}{(q-1)x}.$$

Unlike the classical derivative, which relies on an additive increment $h\to 0$, the q-derivative uses a multiplicative increment by scaling the input. This gives rise to a discrete-like calculus that becomes classical in the limit $q\to 1$:

$$\lim_{q\to 1}D_q\,\mathrm{f}(x)=\mathrm{f}'(x).$$

The q-derivative provides a window into **discrete versions** of continuous phenomena, playing a critical role in **q-series**, **quantum mechanics**, and **non-commutative geometry**. It exemplifies how foundational ideas — such as the derivative — can be reframed and generalised when the underlying assumptions (like continuity or limits) are re-examined.

(a) For $f(x)=x^n$, show that $\lim_{q\to 1} D_q f(x)=f'(x)$. **[3]**

(b) Using the definition $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, show that $\lim_{h\to 0}\frac{\ln(1+h)}{h}=1$. **[3]**

[You may assume that $\ln\left(\lim_{h\to 0} f(h)\right)=\lim_{h\to 0}\ln\left(f(h)\right)$ when f(h) is continuous.]

(c) Using the result in part (b), show that $\lim_{q\to 1} D_q f(x)=f'(x)$ for $f(x)=\ln x$. **[3]**

(d) The product rule from the first principle of differentiation is

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x).$$

(i) Show that for q-derivative, the product rule is

$D_q\left[f(x)g(x)\right]=f(qx)D_q g(x)+g(x)D_q f(x)$ or $f(x)D_q g(x)+g(qx)D_q f(x)$ **[4]**

(ii) State a difference between the product rule from the first principle and from the q-derivative. **[1]**

(e) The quotient rule from the first principle of differentiation is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

(i) Find the quotient rule for the q-derivative. **[4]**

(ii) Comment on the result in part (i) as $q\to 1$, showing clearly your reasoning. **[2]**

[You may assume that $\lim_{x\to a} f(x)g(x)=\lim_{x\to a} f(x)\lim_{x\to a} g(x)$ and $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

when both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and are finite.]