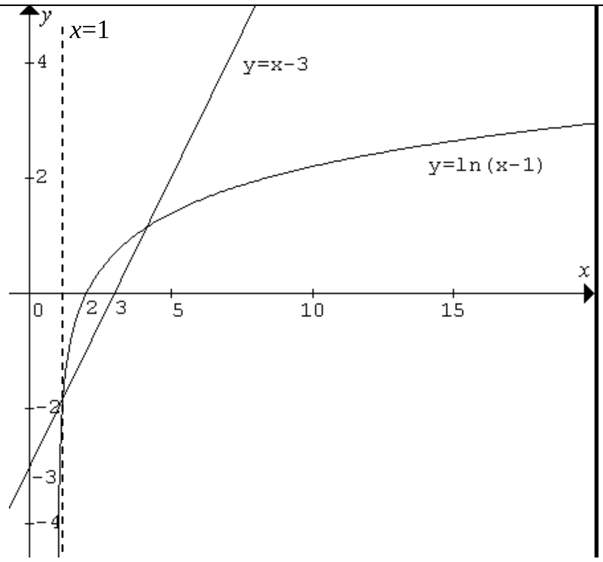
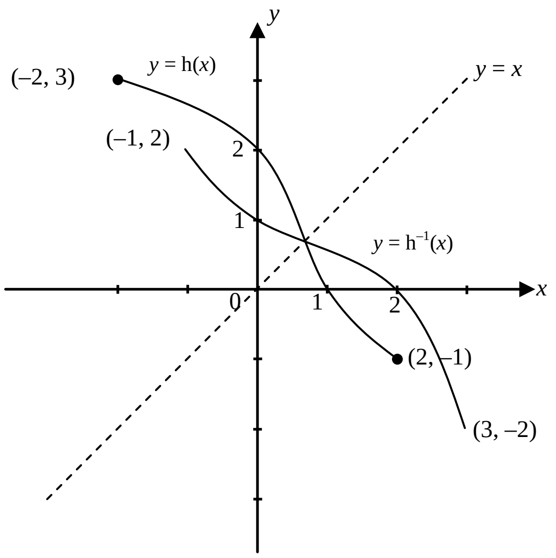
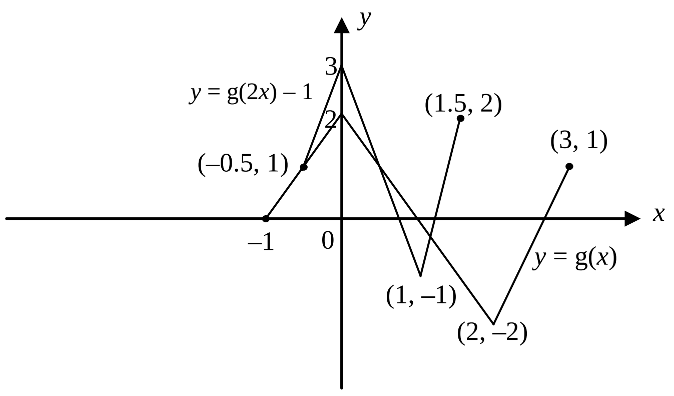
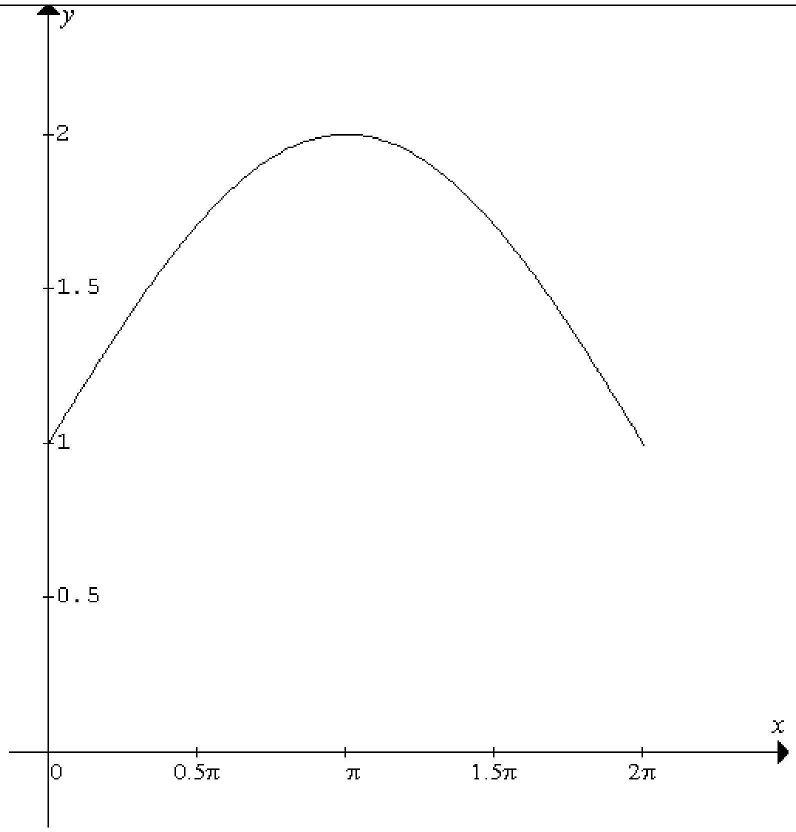
2015 Sec 3 IM2 EOY Mark Scheme

1a	$(6 - x)^2 > x$ $36 - 12x + x^2 > x$ $x^2 - 13x + 36 > 0$ $(x - 9)(x - 4) > 0$ $\therefore\ x < 4$ or $x > 9$
1b	$\alpha + \beta = 3,\ \alpha\beta = 5$ $\alpha^2 - \frac{1}{\beta} + \beta^2 - \frac{1}{\alpha} = \alpha^2 + \beta^2 - \frac{1}{\alpha} - \frac{1}{\beta}$ $= (\alpha + \beta)^2 - 2\alpha\beta - \left(\frac{\alpha + \beta}{\alpha\beta}\right)$ $= 9 - 10 - \left(\frac{3}{5}\right)$ $= -\frac{8}{5}$ $\left(\alpha^2 - \frac{1}{\beta}\right)\left(\beta^2 - \frac{1}{\alpha}\right) = (\alpha\beta)^2 - \alpha - \beta + \left(\frac{1}{\alpha\beta}\right)$ $= (\alpha\beta)^2 - (\alpha + \beta) + \left(\frac{1}{\alpha\beta}\right)$ $= 25 - 3 + \left(\frac{1}{5}\right)$ $= \frac{111}{5}$ New equation: $x^2 + \frac{8}{5}x + \frac{111}{5} = 0$ $5x^2 + 8x + 111 = 0$
2a	$e^{2x} - 7(e^x) - 2 = 0$ $(e^x)^2 - 7(e^x) - 2 = 0$ $e^x = \frac{7 \pm \sqrt{49 - 4(1)(-2)}}{2}$ $e^x = \frac{7 + \sqrt{57}}{2}$ or $\frac{7 - \sqrt{57}}{2}$ (NA) $\therefore\ x = \ln\frac{7 + \sqrt{57}}{2}$ or 1.98
2b	$\frac{\log_3 y}{\log_3 9} - 2 = \log_3 y$ $\frac{1}{2}\log_3 y - 2 = \log_3 y$ $\frac{1}{2}\log_3 y = -2$ $\log_3 y = -4$ $\therefore\ y = \frac{1}{81}$

3a	$2\mathrm{f}(x-5) = 2x^2 - 16x + 32$ $\mathrm{f}(x-5) = x^2 - 8x + 16$ $\qquad = (x-4)^2$ $\qquad = (x+1-5)^2$ $\therefore\ \mathrm{f}(x) = (x+1)^2$ *OR* Reverse 2nd transformation: $y = (2x^2 - 16x + 32) \div 2$ $\ \ = x^2 - 8x + 16$ Reverse 1st transformation: $y = (x+5)^2 - 8(x+5) + 16$ $\ \ = x^2 + 2x + 1$ $\ \ = (x+1)^2$
3bi	
3bii	$x = \dfrac{1}{\mathrm{e}^{3-x}} + 1$ $x - 1 = \mathrm{e}^{x-3}$ $\ln(x-1) = x - 3$ Draw the line $y = x - 3$ $\therefore$ number of solutions is 2
4	$27^x \div 3^y = 9 \Rightarrow 3^{3x} \div 3^y = 3^2 \Rightarrow 3x - y = 2$ ----(1) $2^{2x} \times 4^{1-y} = 64 \Rightarrow 2^{2x} \times 2^{2-2y} = 2^6 \Rightarrow x - y = 2$ ----(2) (or $2x - 2y = 4$) (1) – (2): $x = 0; y = -2$
5i	84000
5ii	$84000\mathrm{e}^{3k} = 42000$ $\mathrm{e}^{3k} = ½$ $3k = \ln ½$ $\therefore\ k = \dfrac{1}{3}\ln\dfrac{1}{2}$ (or –0.231)
5iii	$84000\ \mathrm{e}^{kt} = 16800$

	$e^{kt} = 0.2$ $kt = \ln 0.2$ $\therefore\ t = 6.97$ (3sf)
6a	$(x + 4)^2 + (x + p)^2 = 8$ $2x^2 + (8 + 2p)x + (8 + p^2) = 0$ Eqn has real roots ➔ Discriminant ≥ 0 $(8 + 2p)^2 - 4(2)(8 + p^2) \geq 0$ $-4p^2 + 32p \geq 0$ $p^2 - 8p \leq 0$ OR $-p^2 + 8p \geq 0$ $p(p - 8) \leq 0$ $p(8 - p) \geq 0$ $\therefore\ 0 \leq p \leq 8$
b	$x^2 - 4x + c$ $= x^2 - 4x + 4 + c - 4$ $= (x - 2)^2 + (c - 4)$ $c - 4 = 3$ $\therefore\ c = 7$
7a	
7b	
8i	Max = 0, Min = – 2
8ii	4π

8iii	1
8iv	$\sin\left(\frac{x}{2}\right)=-1$ $\frac{x}{2}=\frac{3\pi}{2}$ $\therefore\ x = 3\pi$
8v	
9ai **ii** **iii**	$\tan\theta$ $-\sin\theta$ $\sin\theta$
9b	$\tan x + 2(1+\tan^2 x) - 5 = 0$ $2\tan^2 x + \tan x - 3 = 0$ $(\tan x - 1)(2\tan x + 3) = 0$ $\tan x = 1$ or $\tan x = -\frac{3}{2}$ Ref. $\angle = \frac{\pi}{4}$ or 0.98279 $\therefore\ x = \frac{\pi}{4}, \frac{5\pi}{4}$, 2.15 or 5.30
10a	LHS $= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$ $= \frac{1-\cos A}{\sin A}$

	$= \dfrac{1-\cos A}{\sin A}\times\dfrac{1+\cos A}{1+\cos A}$ $= \dfrac{1-\cos^2 A}{\sin A(1+\cos A)}$ $= \dfrac{\sin^2 A}{\sin A(1+\cos A)}$ $= \dfrac{\sin A}{1+\cos A}$ = RHS (shown)
10bi	A in 2nd quadrant, hyp = 5 $\cos A = -\dfrac{4}{5}$
10bii	B in 3rd quadrant, opp = 12 $\tan B = \dfrac{12}{5}$
10biii	$\cos ec A \sec B = \dfrac{1}{\sin A}\times\dfrac{1}{\cos B}$ $= \left(\dfrac{5}{3}\right)\left(-\dfrac{13}{5}\right)$ $= -\dfrac{13}{3}$ or $-4\dfrac{1}{3}$
11i	$\dfrac{3}{2x+1} = x$ $2x^2 + x = 3$ $2x^2 + x - 3 = 0$ $(2x + 3)(x - 1) = 0$ $\therefore x = -\dfrac{3}{2}$ or 1
11ii	$f^2(x) = f(\dfrac{3}{2x+1})$ $= \dfrac{3}{2\left(\dfrac{3}{2x+1}\right)+1}$ $= \dfrac{3}{\dfrac{6+2x+1}{2x+1}}$ $= \dfrac{3(2x+1)}{2x+7}$ or $\dfrac{6x+3}{2x+7}$ $f^2 : x \mapsto \dfrac{3(2x+1)}{2x+7}$ where $x \neq -\dfrac{7}{2}, x \neq -\dfrac{1}{2}$
11iii	$g(x) = x^2 - 3x + 1$

<table>
<tr><td></td><td>
$= \left(x-\frac{3}{2}\right)^2 + 1 - \frac{9}{4}$
$= \left(x-\frac{3}{2}\right)^2 - \frac{5}{4}$
Turning point is $\left(1\frac{1}{2}, -1\frac{1}{4}\right)$
The range of g is $g(x) \geq -1\frac{1}{4}$
</td></tr>
<tr><td>11iv</td><td>
Max. $k = 1\frac{1}{2}$
<table>
<tr><td>Let $y = g(x)$</td><td>Let $y = g(x)$</td><td>Let $g^{-1}(x) = y$</td></tr>
<tr><td>$y = \left(x-\frac{3}{2}\right)^2 - \frac{5}{4}$</td><td>$y = x^2 - 3x + 1$</td><td>$x = g(y)$</td></tr>
<tr><td>$\left(x-\frac{3}{2}\right)^2 = y + \frac{5}{4}$</td><td>$x^2 - 3x + 1 - y = 0$</td><td>$x = \left(y-\frac{3}{2}\right)^2 - \frac{5}{4}$</td></tr>
<tr><td>$x - \frac{3}{2} = \pm\sqrt{y + \frac{5}{4}}$</td><td>$x = \frac{3 \pm \sqrt{9 - 4(1-y)}}{2}$</td><td>$\left(y-\frac{3}{2}\right)^2 = x + \frac{5}{4}$</td></tr>
<tr><td>$x = \pm\sqrt{y + \frac{5}{4}} + \frac{3}{2}$</td><td>$x = \frac{3 \pm \sqrt{5+4y}}{2}$</td><td>$y - \frac{3}{2} = \pm\sqrt{x + \frac{5}{4}}$</td></tr>
<tr><td>$g^{-1}(x) = \pm\sqrt{x + \frac{5}{4}} + \frac{3}{2}$</td><td>$g^{-1}(x) = \frac{3 \pm \sqrt{5+4x}}{2}$</td><td>$y = \pm\sqrt{x + \frac{5}{4}} + \frac{3}{2}$</td></tr>
</table>
Since $g^{-1}(x) \leq 1\frac{1}{2}$,
$g^{-1}: x \mapsto -\sqrt{x + \frac{5}{4}} + \frac{3}{2}\ , x \geq -\frac{5}{4}$
or $\frac{3 - \sqrt{5+4x}}{2}$
</td></tr>
<tr><td></td><td>
Bonus qn
$\frac{1}{\log_{ab} b} - \frac{1}{\log_{ab} a} = \log_b ab - \log_a ab$
$= (\log_b a + 1) - (\log_a b + 1)$
$= \log_b a - \log_a b$
$= \frac{1}{\log_a b} - \frac{1}{\log_b a}$
$= \sqrt{\left(\frac{1}{\log_a b} - \frac{1}{\log_b a}\right)^2}$
$= \sqrt{\left(\frac{1}{\log_a b} + \frac{1}{\log_b a}\right)^2 - 4\left(\frac{1}{\log_a b}\right)\left(\frac{1}{\log_b a}\right)}$
$= \sqrt{293 - 4}$
$= 17$
</td></tr>
</table>