Raffles Institution
H2 Mathematics (9758)
Solution for 2019 A-Level Paper 1

Question 1

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
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The coefficient of the cubic function are all real. Since $2+\mathrm{i}$ is a root of f (z) = 0, then $(2+\mathrm{i})^* = 2-\mathrm{i}$ is also a root.

$$\mathrm{f}(z)=0$$
$$\left(z-(-3)\right)\left(z-(2+\mathrm{i})\right)\left(z-(2-\mathrm{i})\right)=0$$
$$(z+3)\left(z^2-4z+5\right)=0$$
$$z^3-z^2-7z+15=0$$
$$az^3+bz^2+cz+d=a\left(z^3+\frac{b}{a}z^2+\frac{c}{a}z+\frac{d}{a}\right)$$
$$b=-a,\ c=-7a,\ d=15a$$

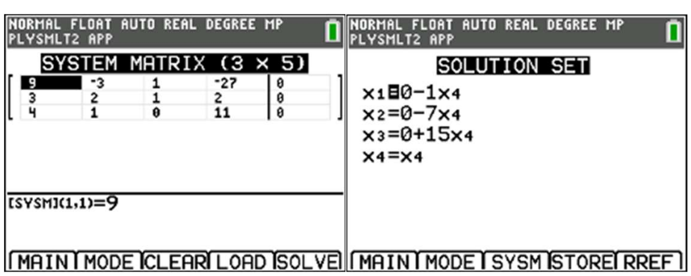
Alternatively,

With the given roots apply Factor Theorem.

$$\mathrm{f}(-3)=0 \Rightarrow -27a+9b-3c+d=0 \qquad (1)$$

$$\mathrm{f}(2+\mathrm{i})=0 \Rightarrow (2+11\mathrm{i})a+(3+4\mathrm{i})b+(2+\mathrm{i})c+d=0$$
$$2a+3b+2c+d=0 \qquad (2)$$
$$11a+4b+c=0 \qquad (3)$$

From (1), (2), (3) we form a system of linear equations with last column the coefficient of *a*.



As before, $b=-a,\ c=-7a,\ d=15a$

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Question 2

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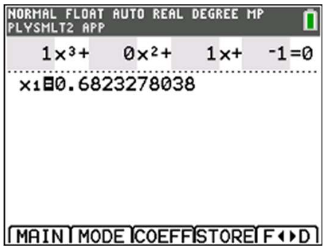
<tr><td>(i)</td><td>

$x^3+x-1=0$

Using GC, Polysimult2

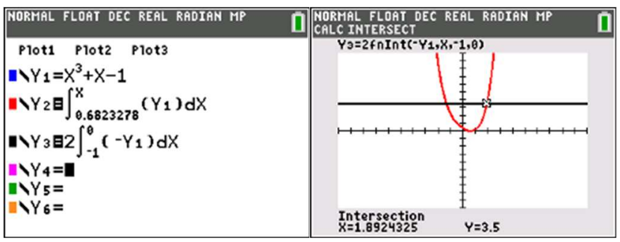


$0.68233 \approx 0.682$

</td><td>Do note the GC skills required in this question.</td></tr>
<tr><td>(ii)</td><td>

$\int_a^b \left(x^3+x-1\right)\mathrm{d}x = 2\int_{-1}^{0} 0-\left(x^3+x-1\right)\mathrm{d}x$

We can solve with GC, noting that $b>a$.



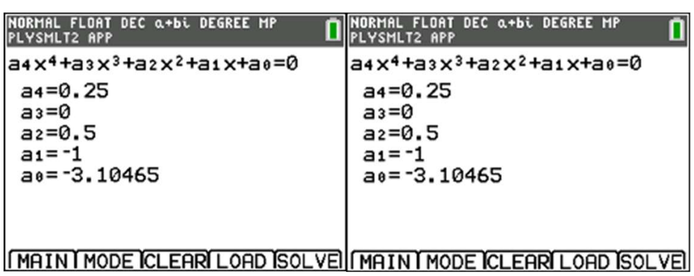
$b=1.89243\approx 1.892$

Alternatively,

$\int_a^b \left(x^3+x-1\right)\mathrm{d}x = 2\int_{-1}^{0} 0-\left(x^3+x-1\right)\mathrm{d}x$

$\left(\frac{x^4}{4}+\frac{x^2}{2}-x\right)-\left(\frac{0.68233^4}{4}+\frac{0.68233^2}{2}-0.68233\right)=3.5$

$\frac{x^4}{4}+\frac{x^2}{2}-x-3.10465=0$



$b=1.89243\approx 1.892$

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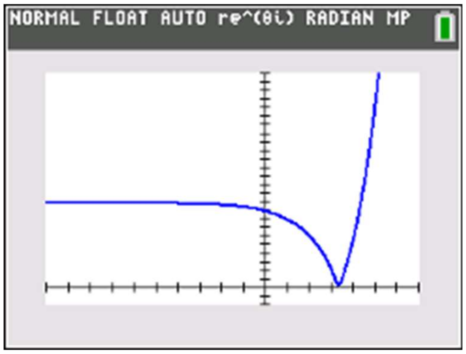
Question 3

No.	Suggested Solution	Remarks for Student
(i)	Note that $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ $\mathrm{f}(x) = 2x^3 - 6x^2 + 6x - 12$ $= 2(x^3 - 3x^2 + 3x - 1) - 10$ $= 2(x-1)^3 - 10$ $= 2\{(x-1)^3 - 5\}$ $p = 2, q = -1, r = -5$	
(ii)	Sequence of 1. Translation of 1 unit in the positive x-direction 2. Translation of 5 units in the negative y-direction 3. Scale by factor 2 parallel to the y-axis.	

Question 4

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(i)</td><td>

$x=0,\ y=\left|2^0-10\right|=9$
$y=0,\ 2^x=10 \Rightarrow x=\dfrac{\ln 10}{\ln 2}$ or $\dfrac{1}{\lg 2}$</td><td></td></tr>
<tr><td>(ii)</td><td>Solving $\left|2^x-10\right|=6$
$2^x-10=6$ or $2^x-10=-6$
$x=\dfrac{\ln 16}{\ln 2}=\dfrac{4\ln 2}{\ln 2}=4$ $x=\dfrac{\ln 4}{\ln 2}=2$
$\left|2^x-10\right|\le 6 \Rightarrow 2\le x\le 4$</td><td></td></tr>
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Question 5

No.	**Suggested Solution**	**Remarks for Student**
(i)	$\mathrm{f}(x) = \mathrm{e}^{2x} - 4,\ x \in \mathbb{R}$ $\mathrm{g}(x) = x + 2,\ x \in \mathbb{R}$ $y = \mathrm{e}^{2x} - 4$ $\mathrm{e}^{2x} = y + 4$ $2x = \ln(y+4)$ $x = \frac{1}{2}\ln(y+4)$ $\mathrm{f}^{-1}(x) = \frac{1}{2}\ln(x+4)$ Domain of f^{-1} = Range of f $= (-4, \infty)$	Note that we use “round brackets” here.
(ii)	$\mathrm{fg}(x) = 5$ $\mathrm{f}^{-1}(\mathrm{fg}(x)) = \mathrm{f}^{-1}(5)$ $\mathrm{g}(x) = \mathrm{f}^{-1}(5)$ $x + 2 = \frac{1}{2}\ln(5+4)$ $x = \ln 3 - 2$	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	$\frac{1}{4r^2-1}=\frac{1}{(2r-1)(2r+1)}=\frac{1}{2(2r-1)}-\frac{1}{2(2r+1)}$ $\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{(2r-1)}-\frac{1}{(2r+1)}\right)$ $=\frac{1}{2}\left(\frac{1}{1}-\cancel{\frac{1}{3}}\right.$ $+\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}$ $+\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}$ $\ldots$ $+\cancel{\frac{1}{(2n-3)}}-\cancel{\frac{1}{(2n-1)}}$ $\left.+\cancel{\frac{1}{(2n-1)}}-\frac{1}{(2n+1)}\right)$ $=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$	
(ii)	$\sum_{r=11}^{\infty}\frac{1}{4r^2-1}=\sum_{r=1}^{\infty}\frac{1}{4r^2-1}-\sum_{r=1}^{10}\frac{1}{4r^2-1}$ $=\frac{1}{2}-\frac{1}{2}\left(1-\frac{1}{2(10)+1}\right)$ $=\frac{1}{42}$	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$y = xe^{-x}$ $\dfrac{dy}{dx} = e^{-x} - xe^{-x}$ $x = 1,\ y = e^{-1},\ \dfrac{dy}{dx} = e^{-1} - e^{-1} = 0$ Equation of tangent: $y = e^{-1}$ $x = -1,\ y = -e,\ \dfrac{dy}{dx} = e + e = 2e$ Equation of tangent: $y + e = 2e(x+1)$ $y = 2ex + e$	
(ii)	$\tan\theta = 2e$ $\theta = 79.6^{\circ}$	Note that one of the tangents is a horizontal line. So the gradient of the other line allows us to get the angle directly.

Question 8

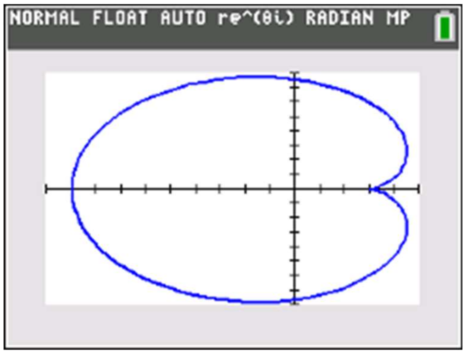
No.	Suggested Solution	Remarks for Student
(a)	$a\left(2^{k-1}\right)=\frac{64}{2}\left(2a+(64-1)(2a)\right)$ $2^{k-1}=32(2+126)$ $2^{k-1}=2^5\left(2^7\right)$ $k=13$	
(b)	Note that $f\neq 0$. If $r=1$, then sum of first 4 terms $=4f\neq 0$ Thus, $r\neq 1$. $\frac{f\left(1-r^4\right)}{1-r}=0$ $r^4=1\Rightarrow r=-1$ or $r=1$ (rejected) Thus, $r=-1$, $f\in\mathbb{R}\setminus\{0\}$ $S_n=\begin{cases}0, & n \text{ even}\\ f, & n \text{ odd}\end{cases}$	
(iii)	Let a be the first term. $\frac{4}{2}(2a+3d)=14$ $2a+3d=7$ ...(1) $a(a+d)(a+2d)(a+3d)=0$ $a=0$ or $-d$ or $-2d$ or $-3d$ Given $a<0$, Possible $d=-a$ or $-\frac{a}{2}$ or $-\frac{a}{3}$ Into (1), only $d=-a$ gives $a<0$ $a=-7, d=7$ 11th term $=a+10d=63$	

Question 9

No.	Suggested Solution	Remarks for Student
(i)	$w=\cos\theta+\mathrm{i}\sin\theta$	
(a)	$\begin{aligned} w+\frac{1}{w} &=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{\cos\theta+\mathrm{i}\sin\theta} \\ &=\cos\theta+\mathrm{i}\sin\theta+\frac{\cos\theta-\mathrm{i}\sin\theta}{\cos^2\theta+\sin^2\theta} \\ &=\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta \\ &=2\cos\theta \end{aligned}$ Alternatively, $\begin{aligned} w+\frac{1}{w} &=\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{\mathrm{e}^{\mathrm{i}\theta}} \\ &=\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta} \\ &=\left(\cos\theta+\mathrm{i}\sin\theta\right)+\left(\cos\theta-\mathrm{i}\sin\theta\right) \\ &=2\cos\theta \end{aligned}$	
(b)	$\begin{aligned} \frac{w-1}{w+1} &=\frac{w+1-2}{w+1} \\ &=1-\frac{2}{1+\cos\theta+\mathrm{i}\sin\theta} \\ &=1-\frac{2\left(1+\cos\theta-\mathrm{i}\sin\theta\right)}{\left(1+\cos\theta\right)^2+\sin^2\theta} \\ &=1-\frac{2\left(1+\cos\theta\right)-\mathrm{i}\left(2\sin\theta\right)}{2+2\cos\theta} \\ &=\frac{2+2\cos\theta-2\left(1+\cos\theta\right)+\mathrm{i}\left(2\sin\theta\right)}{2+2\cos\theta} \\ &=\frac{\mathrm{i}\left(2\sin\theta\right)}{2+2\cos\theta} \\ &=\frac{\mathrm{i}\left(\sin\theta\right)}{1+\cos\theta} \\ &=\frac{\mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}} \\ &=\mathrm{i}\tan\frac{\theta}{2} \end{aligned}$ Alternatively, $\frac{w-1}{w+1}=\frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}0}}{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}0}}=\frac{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}-\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}})}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}(\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}})}$	

	$= \dfrac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}} = \dfrac{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) - \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) + \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)}$ $= \dfrac{2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$ $= i\tan\dfrac{\theta}{2}$	
(ii)	Let $z = a + ib$ Since $\lvert z\rvert = 1 \Rightarrow a^2 + b^2 = 1$ $\lvert z - 3i\rvert = \lvert a + ib - 3i\rvert = \sqrt{a^2 + (b-3)^2} = \sqrt{a^2 + b^2 - 6b + 9} = \sqrt{10 - 6b}$ $\lvert 1 + 3iz\rvert = \lvert 1 + 3i(a + ib)\rvert = \sqrt{(1-3b)^2 + 9a^2} = \sqrt{1 - 6b + 9b^2 + 9a^2} = \sqrt{10 - 6b}$ $\therefore \left\lvert \dfrac{z - 3i}{1 + 3iz} \right\rvert = 1$ OR Since $\lvert z\rvert = 1 \Rightarrow z = \cos\theta + i\sin\theta$ $\lvert z - 3i\rvert = \lvert \cos\theta + i\sin\theta - 3i\rvert = \sqrt{\cos^2\theta + (\sin\theta - 3)^2} = \sqrt{10 - 6\sin\theta}$ $\lvert 1 + 3iz\rvert = \lvert 1 + 3i\cos\theta - 3\sin\theta\rvert = \sqrt{(1 - 3\sin\theta)^2 + 9\cos^2\theta} = \sqrt{10 - 6\sin\theta}$ $\therefore \left\lvert \dfrac{z - 3i}{1 + 3iz} \right\rvert = 1$ OR $\lvert z\rvert = 1 \Rightarrow \lvert z^*\rvert = 1$ $\left\lvert \dfrac{z - 3i}{1 + 3iz} \right\rvert = \left\lvert \dfrac{z - 3i}{1 + 3iz} \right\rvert \lvert z^*\rvert = \left\lvert \dfrac{zz^* - 3iz^*}{1 + 3iz} \right\rvert = \left\lvert \dfrac{1 - 3iz^*}{1 + 3iz} \right\rvert$ Note that $(1 + 3iz)^* = 1^* + (3iz)^* = 1 + (3i)^* z^* = 1 - 3iz^*$ $\therefore \left\lvert \dfrac{z - 3i}{1 + 3iz} \right\rvert = \left\lvert \dfrac{(1 + 3iz)^*}{1 + 3iz} \right\rvert = 1$	

Question 10

No.	Suggested Solution	Remarks for Student
(i)	Taking a as positive.  Label x-intercepts, $(-3a, 0)$ and $(a, 0)$ Cartesian equation of line of symmetry: $y = 0$	Based on the marks allocation, it should be 1 mark for the curve. Use your GC to help you get the shape out by letting a to be a positive number such as 2, 3 or 4.
(ii)	$y = 0 \Rightarrow a\left(2\sin\theta - \sin 2\theta\right) = 0$ $2\sin\theta - 2\sin\theta\cos\theta = 0$ $2\sin\theta\left(1-\cos\theta\right) = 0$ $\sin\theta = 0$ or $\cos\theta = 1$ $\theta = 0, \pi, 2\pi$ or $\theta = 0, 2\pi$ $\therefore \theta = 0, \pi, 2\pi$	
(iii)	Area of region bounded by x-axis and part of C above x-axis $= \int_{\pi}^{0} y \frac{\mathrm{d}x}{\mathrm{d}\theta}\ \mathrm{d}\theta$ $= \int_{\pi}^{0} a\left(2\sin\theta - \sin 2\theta\right) a\left(-2\sin\theta + 2\sin 2\theta\right)\ \mathrm{d}\theta$ $= \int_{\pi}^{0} -a^2\left(4\sin^2\theta - 2\sin\theta\sin 2\theta - 4\sin\theta\sin 2\theta + 2\sin^2 2\theta\right)\ \mathrm{d}\theta$ $= \int_{0}^{\pi} a^2\left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right)\ \mathrm{d}\theta$	From (ii) "lower limit" $\theta = \pi \Rightarrow x = -3a$ "upper limit" $\theta = 0 \Rightarrow x = a$
(iv)	$\int_{0}^{\pi} a^2\left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right)\ \mathrm{d}\theta$ $= a^2\int_0^{\pi} 4\sin^2\theta\ \mathrm{d}\theta - a^2\int_0^{\pi} 6\sin\theta\sin 2\theta\ \mathrm{d}\theta + a^2\int_0^{\pi} 2\sin^2 2\theta\ \mathrm{d}\theta$ $= 2a^2\int_0^{\pi}\left(1-\cos 2\theta\right)\ \mathrm{d}\theta - a^2\int_0^{\pi} 12\sin^2\theta\cos\theta\ \mathrm{d}\theta + a^2\int_0^{\pi}\left(1-\cos 4\theta\right)\ \mathrm{d}\theta$ $= 2a^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} - 4a^2\left[\sin^3\theta\right]_0^{\pi} + a^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi}$ $= 2a^2\pi - 0 + a^2\pi$ $= 3a^2\pi$ $\therefore$ Total area $= 2 \times 3a^2\pi = 6a^2\pi$ from (i) by symmetry.	

Question 11

No.	Suggested Solution	Remarks for Student
(i) **(a)**	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-16), k>0$ $\int \frac{1}{\theta-16}\ \mathrm{d}\theta=\int -k\ \mathrm{d}t$ $\ln(\theta-16)=-kt+c,$ where c is an arbitrary constant and $\theta>16$ $\theta-16=A\mathrm{e}^{-kt},\ A=\mathrm{e}^{c}$ $\theta=16+A\mathrm{e}^{-kt}$ $t=0,\theta=80\Rightarrow A=64$ $\therefore \theta=16+64\mathrm{e}^{-kt}$ $t=30,\theta=32\Rightarrow 32=16+64\mathrm{e}^{-30k}$ $\mathrm{e}^{-30k}=\frac{1}{4}$ $\mathrm{e}^{-k}=\left(\frac{1}{4}\right)^{\frac{1}{30}}$ $\theta=16+64\left(\mathrm{e}^{-k}\right)^{t}=16+64\left(\frac{1}{4}\right)^{\frac{t}{30}}$	
(b)	$\theta=16+64\left(\frac{1}{4}\right)^{\frac{45}{30}}=24$	
(ii)	$\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{\alpha}{T},\alpha>0$ $\int T\ \mathrm{d}T=\int \alpha\ \mathrm{d}t$ $\frac{1}{2}T^{2}=\alpha t+b,$ where b is an arbitrary constant. $t=0,T=0\Rightarrow b=0$ $t=60,T=1\Rightarrow \alpha=\frac{1}{120}$ $\therefore \frac{1}{2}T^{2}=\frac{t}{120}$, that is, $T^{2}=\frac{t}{60}$ We want $T\geq 3\Rightarrow T^{2}\geq 9$ $\frac{t}{60}\geq 9$ $t\geq 540$ Thus, it would be 540 min (9 hours) after freezing commences.	

Question 12

No.	Suggested Solution	Remarks for Student
(i)	$\overrightarrow{PQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \lambda>0$ and Q lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1$ $\overrightarrow{OQ}=\lambda\begin{pmatrix}-2\\-3\\-6\end{pmatrix}+\begin{pmatrix}2\\2\\4\end{pmatrix}$ $\overrightarrow{OQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1\Rightarrow -2\lambda+2-3\lambda+2-6\lambda+4=1\Rightarrow \lambda=\frac{7}{11}$ Q is $\left(2-\frac{14}{11},2-\frac{21}{11},4-\frac{42}{11}\right)=\left(\frac{8}{11},\frac{1}{11},\frac{2}{11}\right)$ $\overrightarrow{RS}=\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}, \mu>0$ and R lies on $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9$ $\overrightarrow{OR}=\begin{pmatrix}-5\\-6\\-7\end{pmatrix}-\mu\begin{pmatrix}-2\\-3\\-6\end{pmatrix}$ $\overrightarrow{OR}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=-9\Rightarrow -5+2\mu-6+3\mu-7+6\mu=-9\Rightarrow \mu=\frac{9}{11}$ R is $\left(-5+\frac{18}{11},-6+\frac{27}{11},-7+\frac{54}{11}\right)=\left(-\frac{37}{11},-\frac{39}{11},-\frac{23}{11}\right)$	
(ii)	$\cos\theta=\dfrac{\left\|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{4+9+36}\sqrt{3}}=\dfrac{11}{7\sqrt{3}}$ $\overrightarrow{RQ}=\frac{1}{11}\begin{pmatrix}8\\1\\2\end{pmatrix}-\frac{1}{11}\begin{pmatrix}-37\\-39\\-23\end{pmatrix}=\frac{1}{11}\begin{pmatrix}45\\40\\25\end{pmatrix}$ $\cos\beta=\dfrac{\left\|\begin{pmatrix}45\\40\\25\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}{\sqrt{45^2+40^2+25^2}\sqrt{3}}=\dfrac{110}{\sqrt{12750}}=\dfrac{22}{\sqrt{510}}$	

(iii)	$\dfrac{\left\lvert \overrightarrow{RQ}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{3}}=\dfrac{\frac{1}{11}(110)}{\sqrt{3}}=\dfrac{10}{\sqrt{3}}$	
(iv)	$k=\dfrac{\sin\theta}{\sin\beta}=\sqrt{\dfrac{1-\cos^2\theta}{1-\cos^2\beta}}=\sqrt{\dfrac{1-\left(\frac{11}{7\sqrt{3}}\right)^2}{1-\left(\frac{22}{\sqrt{510}}\right)^2}}=\dfrac{\sqrt{170}}{7}=1.86$ (3 s.f.)	
(v)	Since $0^\circ<\theta<\beta<90^\circ$, we have $0<\sin\theta<\sin\beta<1$. Thus, $0<\dfrac{\sin\theta}{\sin\beta}<1$, that is, $0<k<1$.	