MA2131 Notes

Foundations in Mathematics IIA

Chapters 1 thru 6

This is a compilation of topics taught in MA2131.
Made by LQ

Links of All LQ Notes: LQ Notes Links Document

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Note: This module has since been renamed to MA2133.

MA2131 Chapter 1: Sets

Set language and Notation

A set is a collection of distinct objects.

Sets are denoted by braces ("{...}").

Example: Musical Instruments = {piano, violin, trumpet...}

Defining sets

Sets can be defined in a few ways.

1. Listing the elements: $A = \{2, 3, 5, 7\}$
2. Describing the set: $A = \{$prime numbers less than $10\}$
3. Defining the elements: $A = \{x : x$ is a prime number less than $10\}$

The above sets are all identical as they all describe primes less than 10.

*The colon in $\{x : x \ldots\}$ is read "such that"

Number of elements in a set

$n(A)$ .. refers to the number of DISTINCT elements in set A

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Element of a Set

To write that the element k is in the set A, we write $k \in A$.

To write that the element k is not in the set A, we write $k \notin A$.

For example, given $A = \{1, 1, 2, 3, 5\}$,
We can say $2 \in A$, $3 \in A$, etc.
However, $4 \notin A$. Similarly, $6 \notin A$.

Note: $n(A)$, in this case, is 4 since there are duplicate 1's.

Finite Set

A set S is a finite set when $n(S)$ is a positive, finite integer.

Infinite Set

A set S is a infinite set when it has infinite elements.
(i.e. It isn't a finite set)

Empty Set

A set S is an empty set when it has no elements.
It is denoted $\{\}$ or $\emptyset$.

Equal set

Two sets are equal if they contain the same elements.
$\{1, 3\} = \{3, 1\}$ since they have the same elements
* Duplicates don't count, so $\{1, 2\} = \{1, 2, 2\}$.

Other Symbols

$\mathbb{N} = \{1, 2, 3, \dots\}$ "natural numbers"
$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ "integers"
$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$ "rational numbers"
* $\mathbb{Q}'$ denotes the set of irrational numbers, (i.e. numbers not in $\mathbb{Q}$)

$\subseteq$: $A \subseteq B$ when A is a subset of B. "subset"
$\subset$: $A \subset B$ when $A \subseteq B$ and $A \neq B$. "proper subset"

Intersection and Union of Sets and Venn Diagram

A Venn Diagram can be used to represent sets.

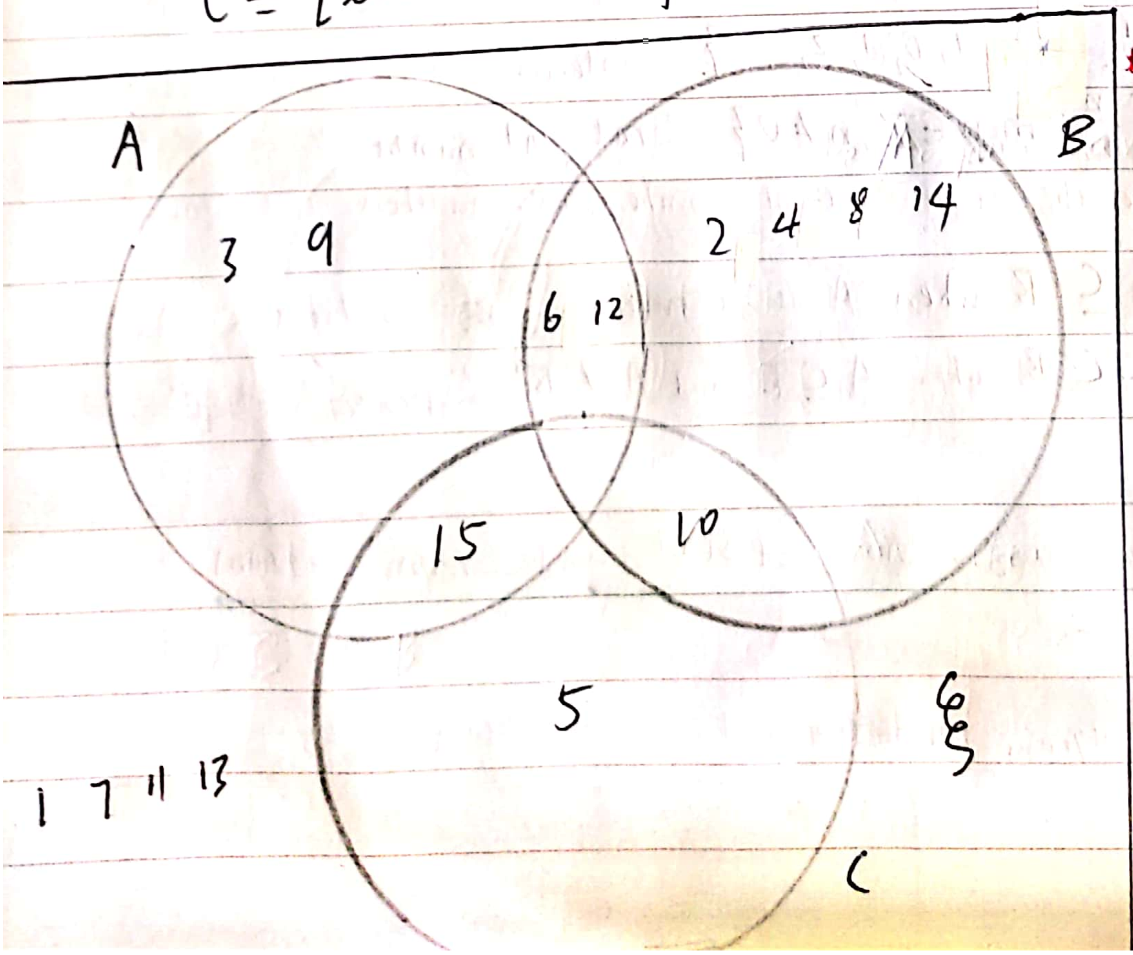
Universal set

The universal set is the set that contains all the elements under consideration in a Venn diagram.

It is denoted by ξ (greek "Xi").

E.g.

$\xi = \{x \in \mathbb{Z};\ 1 \leq x \leq 15\}$
$A = \{x : x \equiv 0 \pmod 3\}$
$B = \{x : x \equiv 0 \pmod 2\}$
$C = \{x : x \equiv 0 \pmod 5\}$



* rectangle is necessary

Complement of a set

The complement of set A is the set containing elements that lie in the universal set, but not in A.

This is denoted A'.

Relative complement of a set

The complement of set B relative to set A is the set of elements in A, but not in B.

This is denoted $A \setminus B$.

Note: $A \setminus B = A \cap B'$

Note: Do NOT use forward-slash (/)!

Intersection of sets

The intersection of sets A and B is the set of elements in both A and B.

This is denoted $A \cap B$.

Union of sets

The union of sets A and B is the set of elements in one or both of A and B.

This is denoted $A \cup B$.

Disjoint Sets

Two sets A and B are disjoint if there are no elements in common in both sets.

This can be expressed "$A \cap B = \varnothing$"

Distributive Law

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

De Morgan's Law

$(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$

Commutative Law

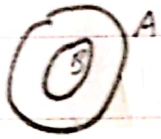
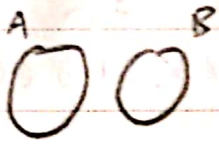
$A \cup B = B \cup A$
$A \cap B = B \cap A$

Associative Law

$[A \cup B] \cup C = A \cup [B \cup C]$
$[A \cap B] \cap C = A \cap [B \cap C]$

* If there is both "$\cup$" and "$\cap$", associative law does NOT apply.

Minimum and Maximum of Union and intersect.

Maximum		
Minimum		
	$A \cup B$	$A \cap B$

Given sets A and B where $n(\overline{A}) \geq n(B)$.

If $n(A) + n(B) \leq n(\xi)$:

- $n(A) \leq n(A \cup B) \leq n(A) + n(B)$
- $0 \leq n(A \cap B) \leq n(B)$

If $n(A) + n(B) > n(\xi)$:

- $n(A) \leq n(A \cup B) \leq n(\xi)$
- $n(A) + n(B) - n(\xi) \leq n(A \cap B) \leq n(B)$

MA2131 Chapter 2: Compound Linear Inequalities

Rules of Inequalities

Transitivity

If $a > b$ and $b > c$, $a > c$.

If $a < b$ and $b < c$, $a < c$.

Reversal

If $a < b$, then $b > a$.

If $a > b$, then $b < a$.

Addition and Subtraction

If $a > b$, then $a + c > b + c$ and $a - c > b - c$.

If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

Multiplication and division

If $a > b$ and $c \in \mathbb{Z}^+$: $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

If $a < b$ and $c \in \mathbb{Z}^+$: $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

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If $a > b$ and $c \in \mathbb{Z}^-$: $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

If $a < b$ and $c \in \mathbb{Z}^-$: $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Additive Inverse

If $a > b$, then $-a < -b$.

If $a < b$, then $-a > -b$.

Multiplicative Inverse

For a, b both positive or both negative:

If $a > b$, $\frac{1}{a} < \frac{1}{b}$.

If $a < b$, $\frac{1}{a} > \frac{1}{b}$.

Interval Notation

Round Brackets "( )" represent open intervals.

Square Brackets "[ ]" represent closed intervals.

$x \in (a, b)$ means $a < x < b$.

$x \in [a, b]$ means $a \leq x \leq b$.

If there is a mix of round and square brackets, it can be called half-open or half-closed.

$x \in (a, b]$ means $a < x \leq b$.

$x \in [a, b)$ means $a \leq x < b$.

NOTE: If ∞ or $-\infty$ is one of the bounds, its bracket MUST be the round bracket.

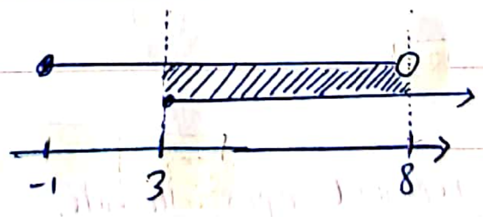
NOTE: If the interval is $[a, \infty)$ or $(\infty, a]$, it is a closed interval.
If the interval is (a, ∞) or $(-\infty, a)$, it is an open interval.

AND and OR

We use a number line.

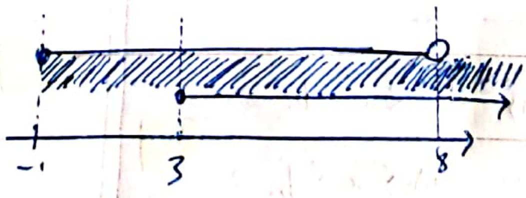
Examples

$\boxed{x \geq 3 \text{ and } -1 \leq x < 8}$



Hence, the answer is $3 \leq x < 8$.

$\boxed{x \geq 3 \text{ or } -1 \leq x < 8}$



Hence, the answer is $x \geq -1$.

MA2131 Chapter 3: Quadratic Function and Inequalities

Factorization

There are many techniques used in factorising.

i) Factoring common factors

$$2ab - 18b^2c = 2b(a - 4bc)$$

ii) Grouping terms with common factors

$$\begin{aligned} 2ax - 2ay - bx + by &= 2a(x-y) - b(x-y) \\ &= (2a-b)(x-y) \end{aligned}$$

iii) Perfect Squares

↳ $(a+b)^2 = a^2 + b^2 + 2ab$

↳ $(a-b)^2 = a^2 + b^2 - 2ab$

iv) Difference of squares

↳ $a^2 - b^2 = (a+b)(a-b)$

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v) Trinomial Faltorization

$2a^2 - a - 3 = (2a-3)(a+1)$

Something to keep in mind is the zero product rule.
(i.e. If $pq = 0$, $p = 0$ or $q = 0$.)

The Quadratic Formula

If $ax^2 + bx + c = 0$, its roots are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* The sign "$\pm$" means "plus or minus".

NOTE: Before using formula, state the values of a, b and c!

Exact Form

Questions might ask you to answer in Exact Form.

This means you should leave non-terminating decimals as-is and not round them.

Examples like π, e, $\sqrt{3}$, $\frac{1}{7}$ should not be rounded. (E.g. $\frac{1}{7} \pm \sqrt{3}$)

Complete the Square

Completing the square refers to rewriting ax^2+bx+c in the form $a(x+p)^2+q$.

We can find the completed square form by:

$$\boxed{a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right]+c}$$

where a, b, c is the polynomial ax^2+bx+c.

Quadratic Graphs

Quadratic equations can be expressed in 3 forms:

- Standard Form: $y = ax^2 + bx + c$
- Factorised Form: $y = a(x-\alpha)(x-\beta)$
- Completed Square Form: $y = a(x-h)^2 + k$

Now when sketching a quadratic graph:

- x-intercepts are α and β.
- x-coordinate of turning point is $\frac{\alpha+\beta}{2}$.
- y-intercept is $(a)(\alpha)(\beta)$.
- y-coordinate of turning point is the equation evaluated at x.

Quadratic Inequalities

We can solve a quadratic inequality $ax^2+bx+c > 0$ or $ax^2+bx+c < 0$, we draw the graph of ax^2+bx+c.

We only need, in this case, to draw out the x-axis and the x intercepts, along with the graph itself.

Example (Example 5, Chapter 3.6 notes)

Find the range of x such that $0 < 7+6x-x^2 \le 12$.

$$\boxed{0 < 7+6x-x^2 \quad \text{AND} \quad 7+6x-x^2 \le 12}$$

$0 < 7+6x-x^2$

$7+6x-x^2 = -(x-7)(x+1)$

$$\boxed{-1 < x < 7}$$

$7+6x-x^2 \le 12$

$-x^2+6x-5 \le 0$

$-x^2+6x-5 = -(x-5)(x-1)$

$$\boxed{x \le 1 \text{ or } x \ge 5}$$

AND

$-1 < x \le 1$ OR $5 \le x < 7$

*Note: Not proper presentation, presentation used to save space.

MA2131 Chapter 4: Congruency and Similarity

Congruency Tests

E.g. SSS Test

Given these information, the two triangles are congruent by SSS.

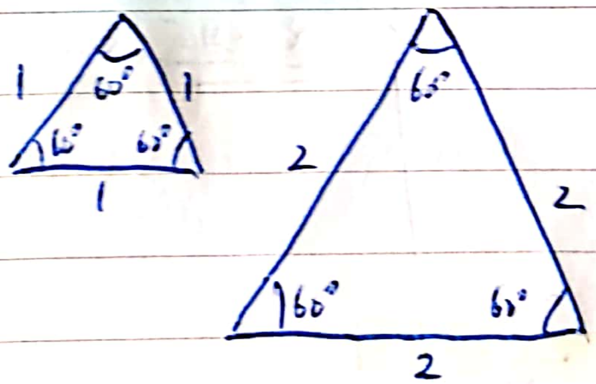
The valid congruency tests are:

1. ASA
2. AAS
3. SAS
4. SSS

Where A means a pair of corr. equal angles,
S means a pair of corr. equal sides.

* Special Case: ASS (SSA) — Only valid when $A = 90^\circ$. Known as RHS test

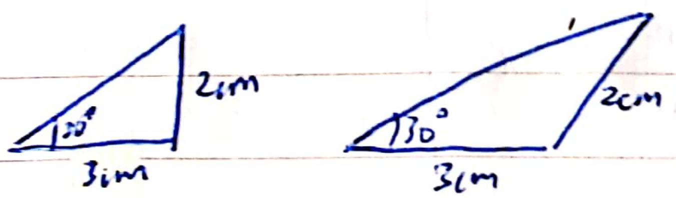
AAA Test (INVALID)



These two have all the same angles, but are not congruent.

Hence, this test is invalid.

ASS Test (INVALID when $A \neq 90^\circ$)



Not to scale exactly, but the two triangles are not congruent and satisfy ASS. (SSA)

Hence this test is invalid.

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Similarity

Two polygons are similar if and only if:

- all the corresponding angles are equal.
- all the corresponding sides are proportional.

scaled by a common factor

Similarity Tests

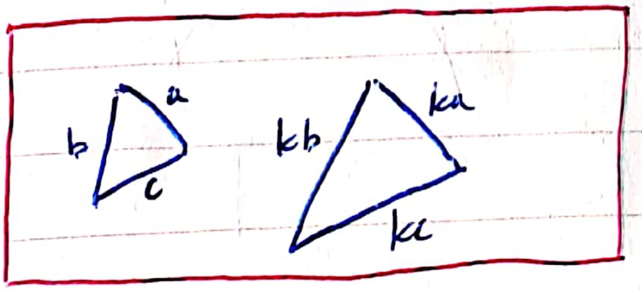
There are 3 tests for similarity.

AAA Property

Two triangles are similar when two of their angles are equal.
(If two angles are equal, the third is also equal).

Proportional Sides

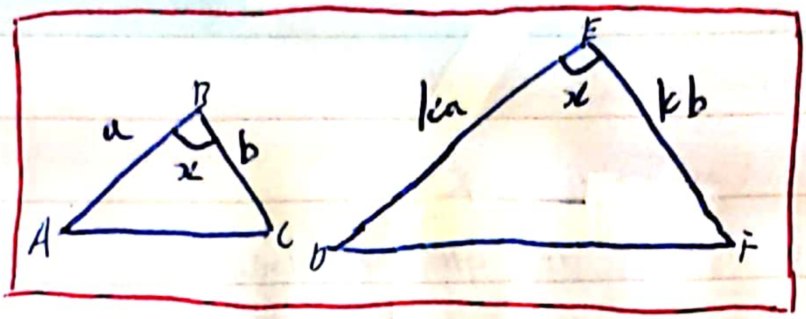
If all the sides of the triangles are proportional, the triangles are equal.



* You just need to present the info. and write, "∴ ... and ... are similar".

NEVER say "SSS" for similarity

Double proportional side and equal included angle



These triangles are similar.

Written as:

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{1}{k}$$

$\angle ABC = \angle DEF$

$\therefore \triangle ABC$ is similar to $\triangle DEF$

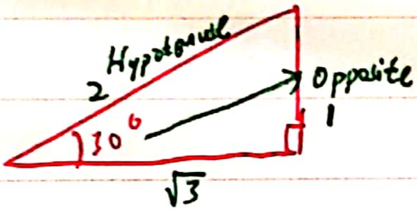
NEVER say "by SAS..." for similarity.

MA2131 Chapter 5: Trigonometry

What are sin, cos and tan?

Sine (written as sin) is the ratio of the length of the opposite side of a triangle, divided by the hypotenuse, in a right angled triangle.

Sin 30°



Hence, $\sin 30^\circ = \frac{Opp.}{Hyp.}$

$= \frac{1}{2}$

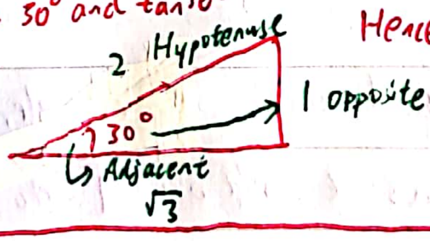
Similarly, cosine (written cos) and tangent (written tan) are also defined by ratios.

$$\sin = \frac{Opp.}{Hyp.}$$

$$\cos = \frac{Adj.}{Hyp.}$$

$$\tan = \frac{Opp.}{Adj.}$$

cos 30° and tan 30°



Hence, $\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\tan 30^\circ = \frac{1}{\sqrt{3}}$

θ: Variable for angles.

Note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

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Important trig values to memorise on $[0°, 90°)$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

These values are important as Trig. graded task does NOT allow calculators.

Inverse Trig Functions

The inverse of sin, $\sin^{-1}$, is defined as the UNIQUE value on $[-90°, 90°]$ so that $\sin(\sin^{-1}\theta) = \theta$.

For example, $\sin 30° = \frac{1}{2}$, so $\sin^{-1}(\frac{1}{2}) = 30°$.

Note the uniqueness of $\sin^{-1}$. For example, $\sin 150° = \frac{1}{2}$ but $\sin^{-1}\frac{1}{2} \neq 150°$ as it isn't on $[-90°, 90°]$.

Similarly, $\cos^{-1}$ is defined as the unique value on $[0°, 180°]$ such that $\cos(\cos^{-1}\theta) = \theta$.
$\tan^{-1}$ is the unique value on $(-90°, 90°)$ such that $\tan(\tan^{-1}\theta) = \theta$.

Trig for obtuse angles

If $\alpha + \beta = 180°$, then

$\sin\alpha = \sin\beta.$
$\cos\alpha = -\cos\beta.$
$\tan\alpha = -\tan\beta.$

For example, when $\alpha = 30°$, $\beta = 150°.$

$\sin\alpha = \sin\beta \Rightarrow \sin 150° = \frac{1}{2}$

Similarly, $\cos 150° = -\frac{\sqrt{3}}{2}$
$\tan 150° = -\frac{1}{\sqrt{3}}.$

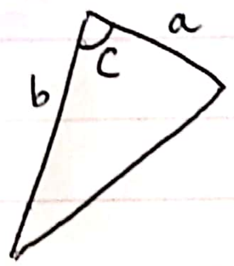
Trig at 0°, 90° and 180°

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
90°	1	0	undefined
180°	0	1	0

*Recall $\tan = \frac{\sin}{\cos}$. Hence, when $\cos = 0$, tan is undefined.

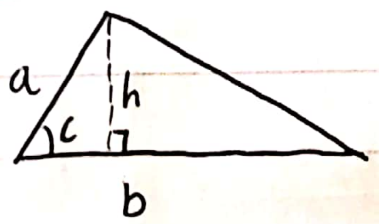
Area of Triangle

When the lengths of two sides of a triangle are given, and its included angle as such:



$$\text{Area} = \frac{1}{2}ab\sin C.$$

Derivation



$h = a\sin C$

$\text{Area of Triangle} = \frac{1}{2}bh$

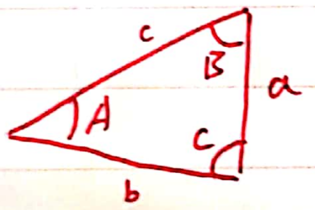
$= \frac{1}{2}ab\sin C.$

Sine Rule

Since the area of a triangle stays constant,

$$\frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A.$$

*Note: It is standard to name sides lowercase, such as a.
The angle opposite the side is the the uppercase.



We can divide each side by $\frac{1}{2}abc$ and obtain:

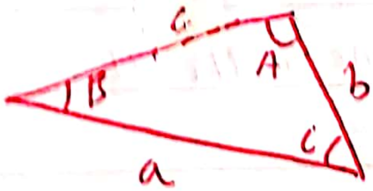
$$\boxed{\begin{gathered}\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a} \\ \text{or equivalently,} \\ \frac{c}{\sin C} = \frac{b}{\sin B} = \frac{a}{\sin A}\end{gathered}}$$

This is the Sine Rule.

Cosine Rule

The cosine rule states:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

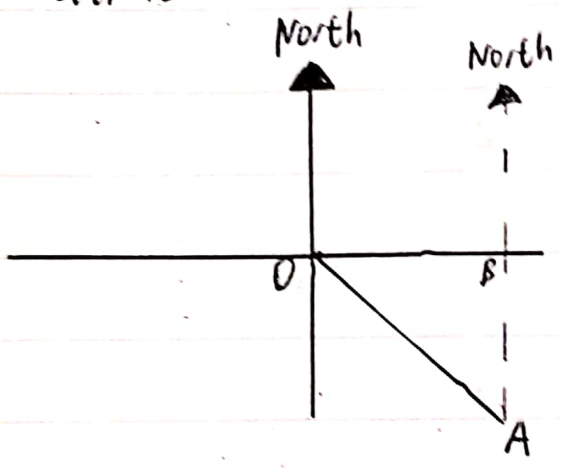


or equivalently,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Bearings

True Bearing is measured from the north in a clockwise direction:



Question: Given ∠BOA is 45°, find the bearing of A from O and O from A.

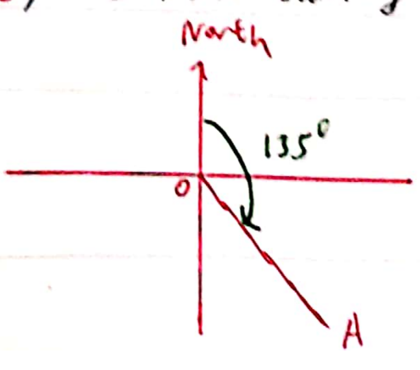
The bearing of A from O is 135°.

The bearing of O from A is 315°.

Note; if the angle is not a whole number, we include the decimal. E.g. The bearing is 126.3°.

if the angle is less than 100°, we add a 0 at the start. E.g. The bearing is 037°.

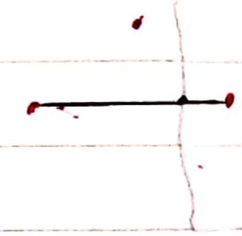
When we say A from O, we mean starting at O and drawing the angle as such:



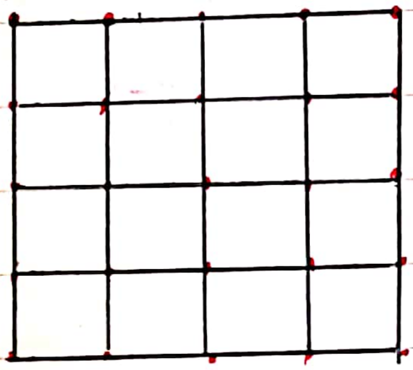
Start from north, then rotate to A.

3 D Problems

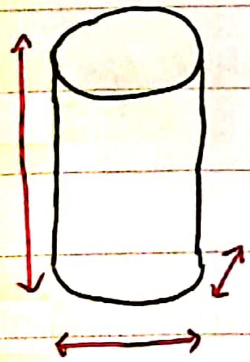
A point has 0 dimensions.
A line has 1 dimension.



A plane is a flat surface and has 2 dimensions.

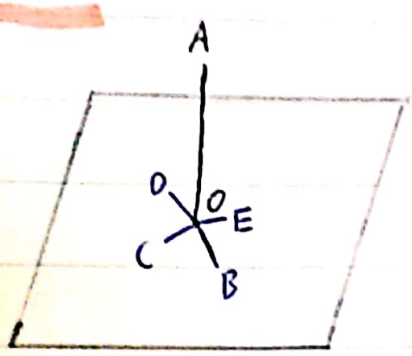


A solid has 3 dimensions.



3 Dimensions.

Normal

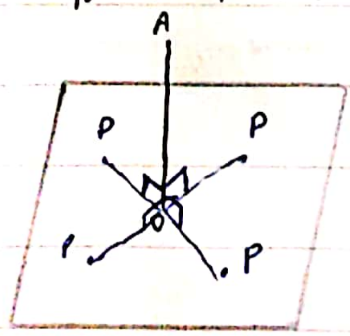


* AO is not on the plane, but a line erected upwards from O on the 3rd dimension.

Consider the plane above.

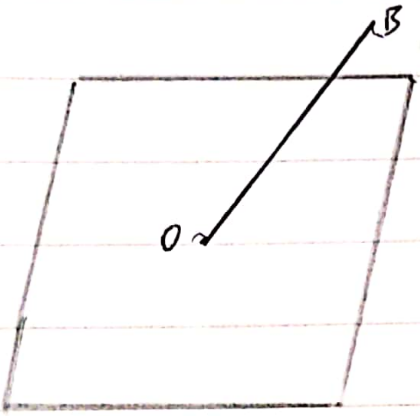
We say AO is the normal to the plane as $\angle AOE$, $\angle AOD$, $\angle AOC$ and $\angle AOB$ are all $90°$.

In general, for any point P on the plane, if $\angle AOP$ is always $90°$, then AO is a normal to the plane.



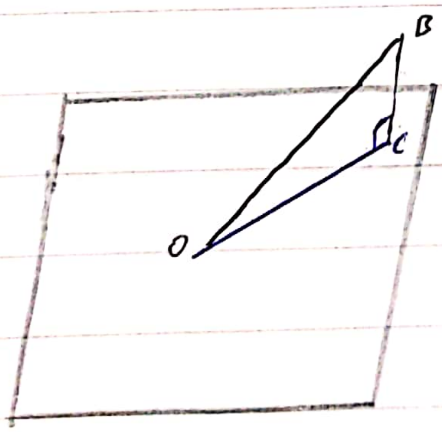
Angle between line and plane.

Consider the following plane.



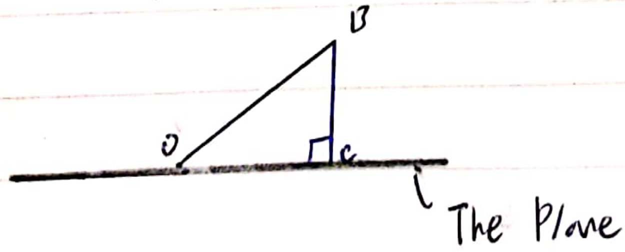
If OB is not normal to the plane, it is inclined at an acute angle to the plane.

* This is by definition. Note that if the angle is obtuse, you can rotate the entire plane 180°. By using this symmetry, you get the acute angle.



We draw the normal CB from the plane, then $\angle BOC$ will be the angle of OB from the plane.

From the side, it looks like:



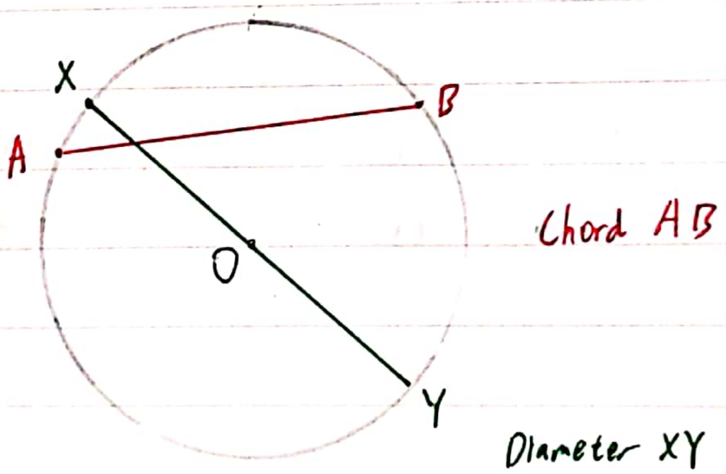
MA2131 Chapter 6: Circle Geometry I

Definitions

A circle is the set of points in a plane that are the same distance from a fixed point called the center (usually denoted "O")

The circumference of a circle is the perimeter of the circle.

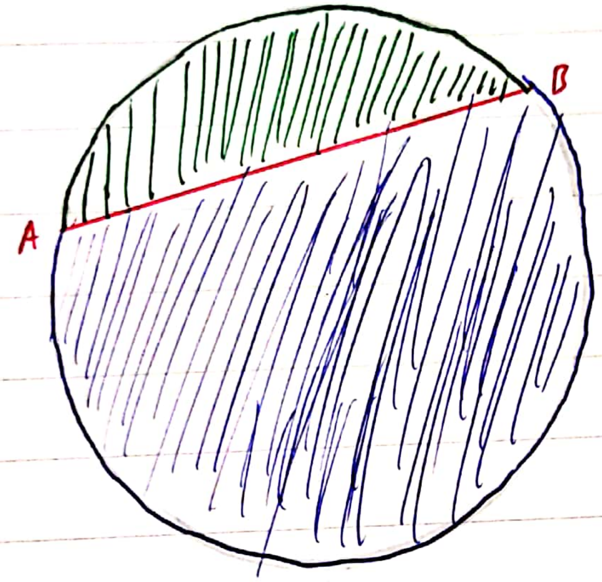
A Chord is a line segment joining two points of a circumference of a circle.



A chord that passes through the center is called a diameter.

4 Jul 2022

A chord separates the circumference and circle itself into two parts.

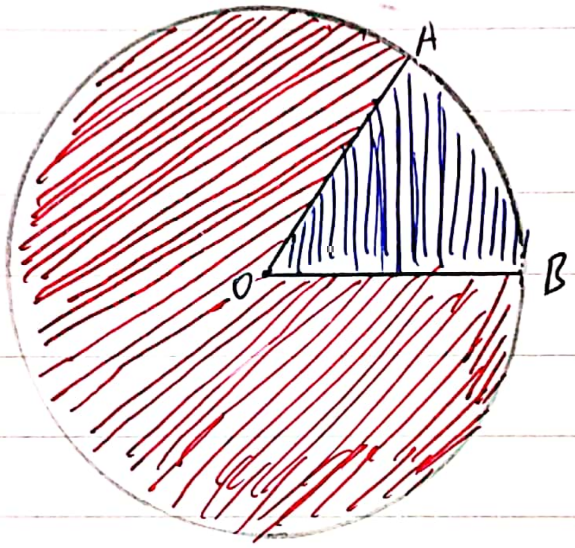


The blue part of the circumference is the major arc.
The green part of the circumference is the minor arc.

The blue part of the circle's area is the major segment
The green part of the circle's area is the minor segment

The radius of a circle is the line segment joining the center to a point on the circumference of the circle.

Two radii (plural of radius) separate the circle into two parts.

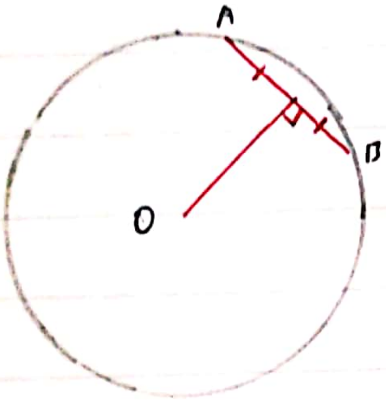


The radii OA and OB separate the circle into the blue sector, called the minor sector, and the red sector, called the major sector..

Circle Properties

NOTE: The following properties don't have an actual name, unless otherwise stated.

Property 1



Abbrev.: ⊥ bisector of chord.

A line draw from O to AB to bisect AB, is perpendicular to AB.

Conversely, if the line is perpendicular to AB, AB is bisected.

Proof

Let the intersection be X.

$OX = OX$ (common side)

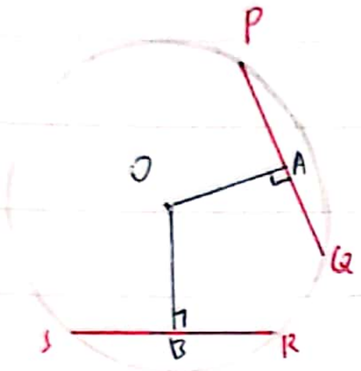
$AO = OB$ (radii)

$\angle OXA = \angle OXB = 90°$

$\therefore$ By RHS, $\triangle OXA \equiv \triangle OXB$

$\therefore AX = XB$ (proven).

Property 2



Abbrev.: Equal chords

If $PQ = RS$, then $OA = OB$.

$QA = \frac{1}{2} PQ$, $BR = \frac{1}{2} RS$ ($\perp$ bisector of chord)

$QA = BR$

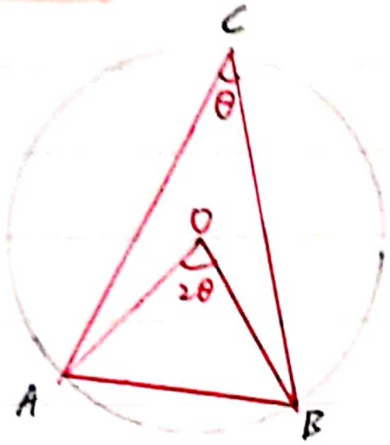
$OQ = OR$ (radii)

$\angle OBR = \angle OAQ = 90°$

$\therefore$ By RHS, $\triangle OBR \equiv \triangle OAQ$

$\therefore OA = OB$ (proven)

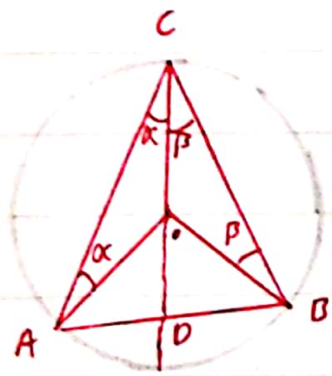
Property 3



Abbrev: ∠ at centre = 2∠ at circ.

Proof

Join OC. Let $\angle ACO = \alpha$ and $\angle BCO = \beta$. Extend OC to D at AB.



AO = OC, so $\triangle ACO$ is isosceles.

Similarly, $\triangle BCO$ is isosceles.

$\therefore \angle CAO = \alpha$, $\angle CBO = \beta$.

$\angle AOD = 2\alpha$ (ext. ∠ = sum of opp. int. ∠s)

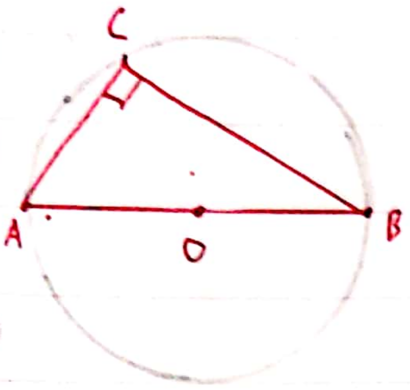
$\angle BOD = 2\beta$ (ext. ∠ = sum of opp. int. ∠s).

$\angle ACB = \alpha + \beta$

$\angle AOB = 2(\alpha + \beta)$.

(proven)

Property 4

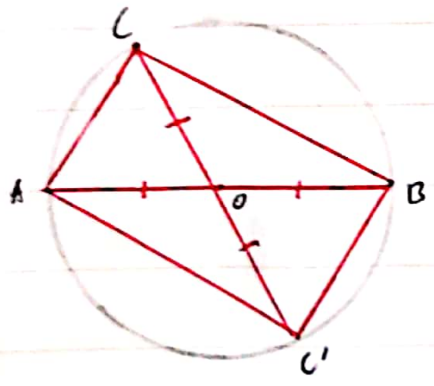


Abbrev.: rt. $\angle$ in semicircle.

Proof

Join OC. Note $AO = OC = OB$ (radii).

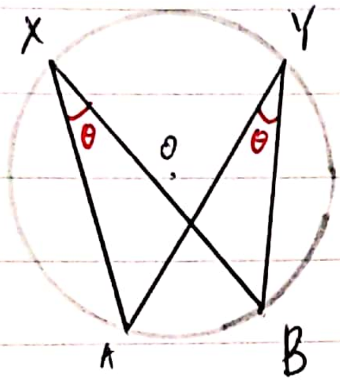
Construct C' such that $BC' = AC$ and $AC // BC'$. Join OC'.



Note $ACBC'$ is a rectangle.

$\therefore \angle ACB = 90^\circ$ (proven).

Property 5



Abbrev: ∠s in same segment

Proof:

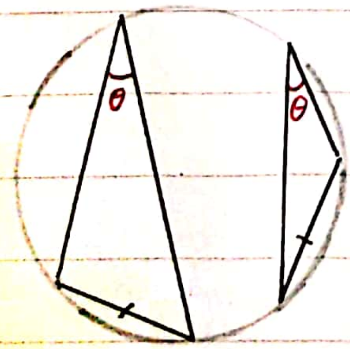
Join OA and OB.

$\angle AOB = 2\angle AXB$ (∠ at centre = 2∠ at circ.)

$\angle AOB = 2\angle AYB$ (∠ at centre = 2∠ at circ.)

$\therefore \angle AXB = \angle AYB$ (proven)

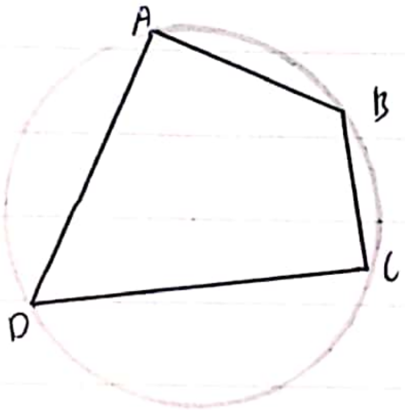
This also works for angles in segments of same length!



Abbrev. ∠s in segments of equal length

Cyclic Quadrilateral

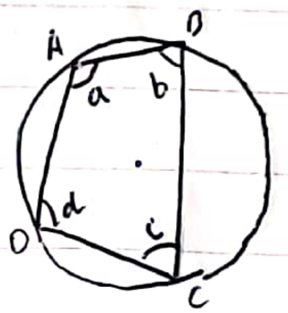
When a quadrilateral has all four vertices on a circle, it is a cyclic quadrilateral.



ABCD is a cyclic quadrilateral.

A, B, C and D are then said to be concyclic.

Property 6



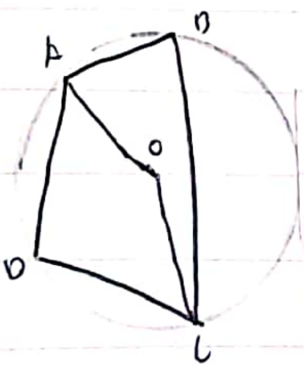
Abbrev. Opp ∠s in cyclic quad.

$\angle a + \angle c = 180°$

$\angle b + \angle d = 180°$.

Proof

Join AO, CO.



Let $\angle ABC = x$, $\angle ADC = y$.

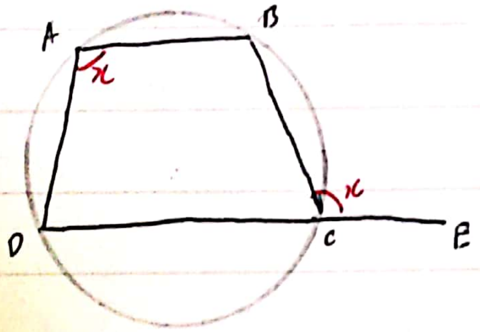
$\angle AOC = 2x$ ($\angle$ at centre $= 2\angle$ at circ.)

Reflex $\angle AOC = 2y$.

$2x + 2y = 360°$ ($\angle$s at a pt.)

$\therefore x + y = 180°$ (proven)

Property 7



Abbrev. Ext. ∠ of cyclic quad.

$\angle DAB = \angle BCE$

Proof

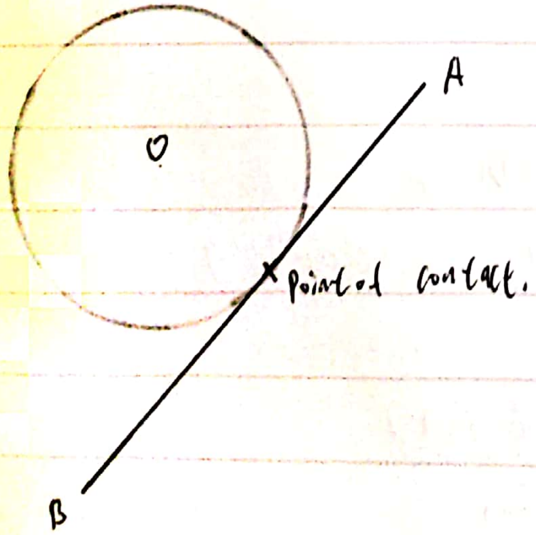
$\angle BCD = 180^\circ - x$ (opp. ∠s of cyclic quad.)

$\angle BCE = 180^\circ - (180^\circ - x)$ (∠s on a line)

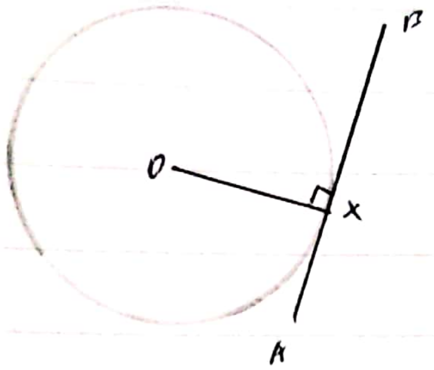
$= x$ (proven)

Tangent Lines

A line AB is said to be tangent to a circle, if and only if, it only touches the circle at one point. That point is then said to be the point of contact or point of tangency.



Property 8

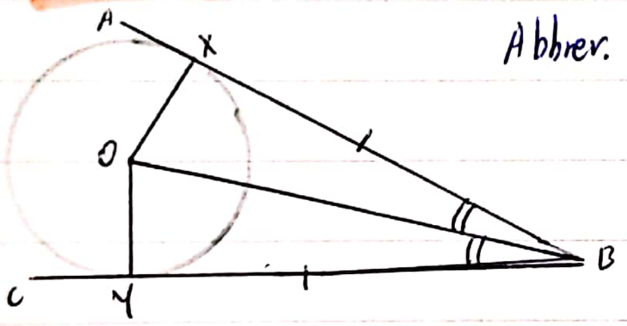


Abbrev.: tang. ⊥ rad.

OX is perpendicular to AB.

This is by definition.

Property 9



Abbrev. tang from ext. pt.

If AB and BC are tangents,

BX = BY, and ∠OBY = ∠OBX.

Proof

Join OB.

∠OYB = 90°, ∠OXB = 90°. (tang. ⊥ rad.)

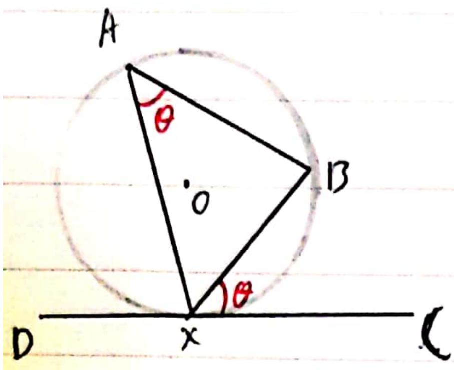
OX = OY (radii)

OB = OB (common side)

∴ △OXB ≅ △OYB (RHS).

Hence, BX = BY and ∠OBX = ∠OBY. (proven)

Property 10 (Alternate Segment Theorem).



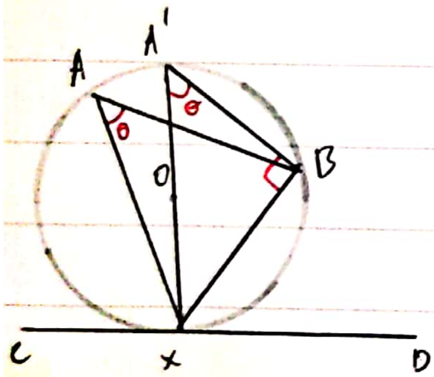
If line CD is tangent to the circle at X,

$\angle AXB = \angle BXC$.

Abbrev. Alt. Seg. Thm. OR

∠s in alt. segment

Proof: Construct A' such that A'X is a diameter.



Join A'B.

$\angle A'BX = 90°$ (rt. ∠ in semicircle)

$\angle XA'B = \angle XAB$ (∠s in same segment)

$\angle A'XB = 180° - 90° - \theta$ (∠s in △).

$= 90° - \theta$

$\angle A'XD = 90°$ (tang. ⊥ rad.)

$\angle BXD = 90° - (90° - \theta)$

$= \theta$ (proven)