NANYANG JUNIOR COLLEGE

JC2 COMMON TEST 2

Higher 2

FURTHER MATHEMATICS **9649/02**

Paper 2 **6th July 2022**

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

NANYANG JUNIOR COLLEGE
Internal Examinations

Section A: Pure Mathematics [50 marks]

1 The curve with equation $y = x + \sin 2x$, for $0 \le x \le 2\pi$, is rotated completely about the y-axis. The volume of the solid of revolution is denoted by V.

(a) Explain with the aid of a sketch, why it is more appropriate to use the method of shells than the method of discs in calculating V. [2]

(b) Verify that V is just under 500 cubic units. [3]

2 **[Do not use a calculator in answering this question]**

Let $\omega = \cos\left(\frac{2\pi}{9}\right) + \mathrm{i}\sin\left(\frac{2\pi}{9}\right)$.

(a) **(i)** Write down, in terms of ω, the roots of $z^9 - 1 = 0$. [1]

(ii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$. [1]

The complex numbers z_1 and z_2 are given by

$$z_1 = \omega + \omega^2 + \omega^3 + \omega^4 \text{ and } z_2 = \omega^{-1} + \omega^{-2} + \omega^{-3} + \omega^{-4}.$$

(b) **(i)** Find the exact value of $z_1 + z_2$, justifying your working. [2]

(ii) Deduce the exact value of

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9}.$$ [3]

(iii) Express $z_1 z_2$ in the form

$$a + b\cos\frac{2\pi}{9} + c\cos\frac{4\pi}{9} + d\cos\frac{6\pi}{9},$$

where a, b, c and d are integers to be determined. [3]

3 The linear transformation $T:\mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -k & 1 & 2 \\ 1 & k & 0 & 0 \\ 2k & k & k & 2k \\ -1 & 0 & -1 & -2 \end{pmatrix}.$$

Let N and R denote the null space and the range space of T respectively.

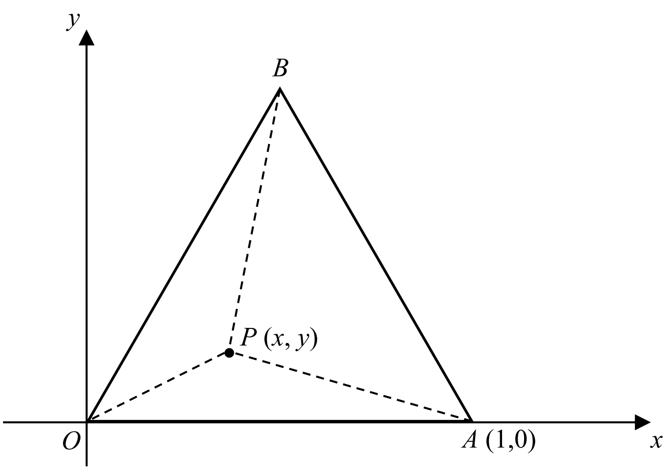
(i) If dim(N) < dim (R), what can you say about the possible value(s) of k ? [2]

It is given that $k = 1$.

(ii) Find a basis for $N \cap R$. [3]

(iii) Let m be the least positive integer such that $\mathbf{A}^m$ represents the zero transformation (i.e. the transformation that maps all elements into the zero element). By first writing $\mathbf{Ax}$ as a linear combination of appropriate basis vectors and considering $\mathbf{A}^2\mathbf{x}$, find m. [4]

4 The diagram below shows an equilateral triangle OAB of side length 1 in a cartesian plane.



The *centroid* of a triangle is the point in the triangle that minimises the sum of squares of distances from the point to each vertex. In the diagram above, the centroid of OAB is the point P that minimises the quantity $S = OP^2 + AP^2 + BP^2$.

(i) By considering $P(x, y) = P(r\cos\theta, r\sin\theta)$, write down S in terms of r and θ, and use differentiation to find the exact coordinates of the centroid of OAB. You should prove that this gives a minimum value of S. [10]

Let G be the midpoint of AB, and denote the centroid of OAB by M.

(ii) Show that O, M and G are collinear. [2]

5 A $n \times n$ matrix is called a Markov matrix if all entries are non-negative and the sum of each column is equal to 1. Prove that a Markov matrix always have an eigenvalue 1. [2]

A caravan company has three locations from which it rents caravan for a day. These locations are Airport, Dandeton and Morningong. The daily proportion of migration of caravans is shown in the table below:

		Rented from		
		Airport	Dandeton	Morningong
Returned to	Airport	0.95	0.02	0.05
	Dandeton	0.03	0.90	0.05
	Morningong	0.02	0.08	0.90

(i) Write down a matrix that represents the migration of caravan. [1]

(ii) Find the characteristic polynomial of the matrix found in **(i)**. [3]

It is given that the initial fraction of caravans at Airport, Dandeton and Morningong are 0.35, 0.35 and 0.3 respectively.

(iii) Find the distribution of caravans after a day. [1]

(iv) Due to space constraints in the Airport, the fraction of caravans in the Airport cannot exceed 0.4. Using eigenvalues and eigenvectors, find the long-term distribution of caravans and state the implications on the business. [7]

Section B: Probability and Statistics [50 marks]

6 Let $X_1, X_2, \ldots, X_n$ be n independent observations of a continuous random variable X which is uniformly distributed between 0 and m. Another random variable W is defined by

$$W = \max\{X_1, X_2, \ldots, X_n\}.$$

(i) Find an expression for $\mathrm{P}(W < w)$ in terms of m, n and w. [2]

(ii) Find the probability density function of W. [1]

(iii) Find $\mathrm{E}(W)$. [2]

7 Most students would probably argue that they failed a test due to careless algebraic errors only. However, considerable errors from misconception and misinterpretation leads to uncertainty to this claim. As such, a study is conducted to compare the differentiated performances of students of those who failed, the average and the high performers, with their total number of conceptual and interpretation errors made in a lecture test on Integration. The following table summarises the finding.

Grades	**Total number of Conceptual and Interpretation Errors made**			**Total**
	≤ 4	**5 – 6**	**≥ 7**	
A	6	0	0	6
B	5	1	0	6
C	3	1	0	4
D	4	6	2	12
E	0	3	1	4
S	1	1	1	3
U	1	6	15	22
Total	20	18	19	57

Carry out a suitable test at 1% significance level to ascertain the students' claim that their performances in the lecture test on Integration is independent of the total number of conceptual and interpretation errors made. [7]

8 The continuous random variable X has probability density function given by

$$\mathrm{f}(x)=\begin{cases}\dfrac{3}{22}(4-x^2) & \text{if } 0\leq x\leq 2,\\ \dfrac{3}{22} & \text{if } 2<x\leq 4,\\ 0 & \text{otherwise.}\end{cases}$$

(i) Given $\mathrm{P}(X>a)=0.1$, find a. [3]

(ii) Given $\mathrm{E}(X)=\dfrac{15}{11}$, find $\mathrm{Var}(X)$. [2]

(iii) 55 independent observations of X are taken. Find the probability that the sum of these observations is between 56 and 78. [2]

9 During recess, Mitch and Megan play a game by tossing a coin. The probability that the coin lands head up is p and tail up is $q=1-p$. Mitch repeatedly tosses the coin until two consecutive heads or two consecutive tails are obtained. If two consecutive heads are obtained, the game ends and Mitch wins. If two consecutive tails are obtained, the game ends and Megan wins.

(i) Find the probability that Mitch wins the game, simplifying your answer. [2]

Let X be the number of tosses needed for a winner to emerge.

(ii) Show that $\mathrm{P}(X=2k)=(pq)^{k-1}(p^2+q^2)$. Find $\mathrm{P}(X=2k+1)$. [4]

(iii) Hence find $\mathrm{E}(X)$ when $p=0.4$. [3]

[You may assume that $\sum_{r=1}^{\infty} rt^{r-1}=(1-t)^{-2}$ for $|t|<1$.]

10 The scores for a group of twelve randomly selected golfers in the first two days of a golf tournament are summarised in the following table:

Golfer	1	2	3	4	5	6	7	8	9	10	11	12
Day 1	69	70	76	69	70	70	69	72	72	70	68	73
Day 2	70	68	74	72	73	75	71	70	76	70	72	71

(i) State, with a reason, whether a two-sample t-test or a paired-sample t-test is more appropriate to justify the claim that the score on Day 2 is more than the score on Day 1. [1]

(ii) Carry out an appropriate test at 10% level of significance to investigate if the golfers scored more points on Day 2 than on Day 1. State a necessary assumption for the test to be valid. [6]

(iii) It was reported that weather conditions affected the performance of the golfers on Day 2. To make the performance on Day 2 comparable to that on Day 1 at 10% level of significance, the tournament organiser decides to lower the score of each player on Day 2 by k, where k is a positive integer. Find the possible values of k. [2]

11 A sky garden is accessible via two lifts A and B which are adjacent to each other. Visitors who wish to visit the sky garden arrive at the lift lobby at random at an average rate of 6 per hour and wait for their turn to take the lift.

(i) Given that the probability that no more than one visitor who arrives at the lift lobby during a period of k minutes is less than 0.2, write down an inequality that must be satisfied by k. Hence find the range of values of k, giving your answer correct to one decimal place. [3]

Each lift carries a maximum of 3 passengers and runs at every 20-minute interval. Assuming that no one is waiting in the lift lobby after the first departure of lift A, find the probability that no one is waiting after the second departure of lift A if

(ii) lift A and lift B leave together at the same time at 20-minute intervals, [1]

(iii) lift A and lift B take turn to leave alternately at 10-minute intervals. [3]

Due to a power failure, both lifts have stopped working and no more visitors arrive to the lift lobby. It is not known when the lifts will be back in operation. There are n visitors waiting in the lift lobby at the time when the queue is closed and visitors can leave the queue independently of others. For each of these visitors, the time from the closure of the queue until the visitor leaves the queue, W minutes, follows an exponential distribution with mean 5 minutes. The time from closure of the queue until the first visitor leaves the queue is denoted by T minutes.

(iv) Find the probability that there is at least one visitor leaving the queue within t minutes after the closure of the queue. Hence, show that T follows an exponential distribution and state the expected time until the first visitor leaves the queue. You are to state the parameters of the distribution clearly. [3]

(v) Explain why the expected time between the first and second visitor leaving the queue is $\dfrac{5}{n-1}$.

[1]

(vi) Find the largest value of n if the expected time for all the visitors to leave the queue is within 10 minutes. [2]

———END OF PAPER———