AM

4E5N Prelim P2

1 (a)

$P(x) = (2x^2 - 3x + 1)\, Q(x) + ax + b$

$= (x-1)(2x-1)\, Q(x) + ax + b$

let $x = 1$.

$a + b = 1$ — (1)

let $x = \frac{1}{2}$.

$\frac{1}{2}a + b = -1$ — (2)

(1) − (2) : $\frac{1}{2}a = 2$

$a = 4$

Sub. $a = 4$ into (1).

$4 + b = 1$

$b = -3$

The remainder is $4x - 3$.

1 (b)

$$(1-\tfrac{1}{4}x)^8 = 1 + {}^8C_1(-\tfrac{1}{4}x) + {}^8C_2(-\tfrac{1}{4}x)^2 + {}^8C_3(-\tfrac{1}{4}x)^3 + \dots$$

$$= 1 - 2x + \tfrac{7}{4}x^2 - \tfrac{7}{8}x^3 + \dots$$

$$(p-3x)(1-\tfrac{1}{4}x)^8$$

$$=(p-3x)(1-2x+\tfrac{7}{4}x^2-\tfrac{7}{8}x^3+\dots).$$

x-term $= p(-2x) - 3x(1)$

$= -2px - 3x$

x^3-term $= p(-\tfrac{7}{8}x^3) + (-3x)(\tfrac{7}{4}x^2)$

$= -\tfrac{7}{8}px^3 - \tfrac{21}{4}x^3$

$\therefore\ -2p - 3 = -\tfrac{7}{8}p - \tfrac{21}{4}$

$2p + 3 = \tfrac{7}{8}p + \tfrac{21}{4}$

$\tfrac{9}{8}p = \tfrac{9}{4}$

$p = 2$ //

2(a) (i) $\tan 2(67.5°) = \dfrac{2\tan 67.5°}{1-\tan^2 67.5°}$

$\tan 135° = \dfrac{2\tan 67.5°}{1-\tan^2 67.5°}$

$-1 = \dfrac{2\tan 67.5°}{1-\tan^2 67.5°}$

$\tan^2 67.5° - 2\tan 67.5° - 1 = 0$

$\tan 67.5° = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$= \dfrac{2 \pm \sqrt{8}}{2}$

$= \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1 + \sqrt{2}$ or $1 - \sqrt{2}$ (rejected).

$\cot 67.5° = \dfrac{1}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}}$ }

$= \dfrac{1-\sqrt{2}}{1-2}$

$= -1 + \sqrt{2}$ where $p = -1$, $q = 1$

a(ii) $\operatorname{cosec}^2 67.5° = 1 + \cot^2 67.5°$

$= 1 + (-1 + \sqrt{2})^2$

$= 1 + (2 - 2\sqrt{2} + 1)$

$= 4 - 2\sqrt{2}$ A1

2(b)

$$\frac{\sin(A+B)}{\sin(A-B)} = \frac{2}{3}$$

$$\frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{2}{3}$$

$$3 \sin A \cos B + 3 \cos A \sin B = 2 \sin A \cos B - 2 \cos A \sin B$$

$$\sin A \cos B = -5 \cos A \sin B$$

$$\frac{\sin A \cos B}{\sin A \cos B} = \frac{-5 \cos A \sin B}{\sin A \cos B}$$

$$1 = -5 \cot A \tan B$$

$$\tan A = -5 \tan B$$

$$5 \tan B + \tan A = 0 \quad \text{(proven)}$$

3(a) Let $f(x) = 2x^3 - 5x^2 + x + 2$.

Let $x = 1$. $f(1) = 2(1)^3 - 5(1)^2 + (1) + 2$

$= 0$

Since $f(1) = 0$, $(x-1)$ is a factor of $2x^3 - 5x^2 + x + 2$.

$$
\begin{array}{r}
2x^2 - 3x - 2 \\
x - 1 \overline{\smash{)}\, 2x^3 - 5x^2 + x + 2} \\
-(2x^3 - 2x^2) \\
\hline
-3x^2 + x \\
-(-3x^2 + 3x) \\
\hline
-2x + 2 \\
-(-2x + 2) \\
\hline
0
\end{array}
$$

* Either long division or compare coeff to obtain Quad. factor.

$\therefore 2x^3 - 5x^2 + x + 2 = (x-1)(2x^2 - 3x - 2)$

$= (x-1)(2x+1)(x-2)$ //

Or

$f(x) = (x-1)(2x^2 + bx - 2)$

comparing coeff. of x: $1 = -b - 2$

$b = -3$

3(b)

$$2(e^{2y-2}) - 5(e^{y-1}) + 2e^{1-y} + 1 = 0$$

$$2(e^{y-1})^2 - 5e^{y-1} + \frac{2}{e^{y-1}} + 1 = 0$$

let $a = e^{y-1}$

$$\therefore 2a^2 - 5a + \frac{2}{a} + 1 = 0$$

$$2a^3 - 5a^2 + a + 2 = 0$$

$$(a-1)(2a+1)(a-2) = 0$$

$a = 1$ or $a = -\frac{1}{2}$ or $a = 2$

Hence, $e^{y-1} = 1$ or $e^{y-1} = 2$ or $e^{y-1} = -\frac{1}{2}$

$y-1 = \ln 1$ $\quad$ $y-1 = \ln 2$ $\quad$ $y-1 = \ln(-\frac{1}{2})$

$y = 1$ $\quad$ $y = 1 + \ln 2$ $\quad$ (no solution)

$= \ln 2e$

4 (a)

$$y = a \sin bx + c.$$

$$c = \frac{6+0}{2}$$

$$= 3$$

$$\frac{2\pi}{b} = 2 \times \left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$$

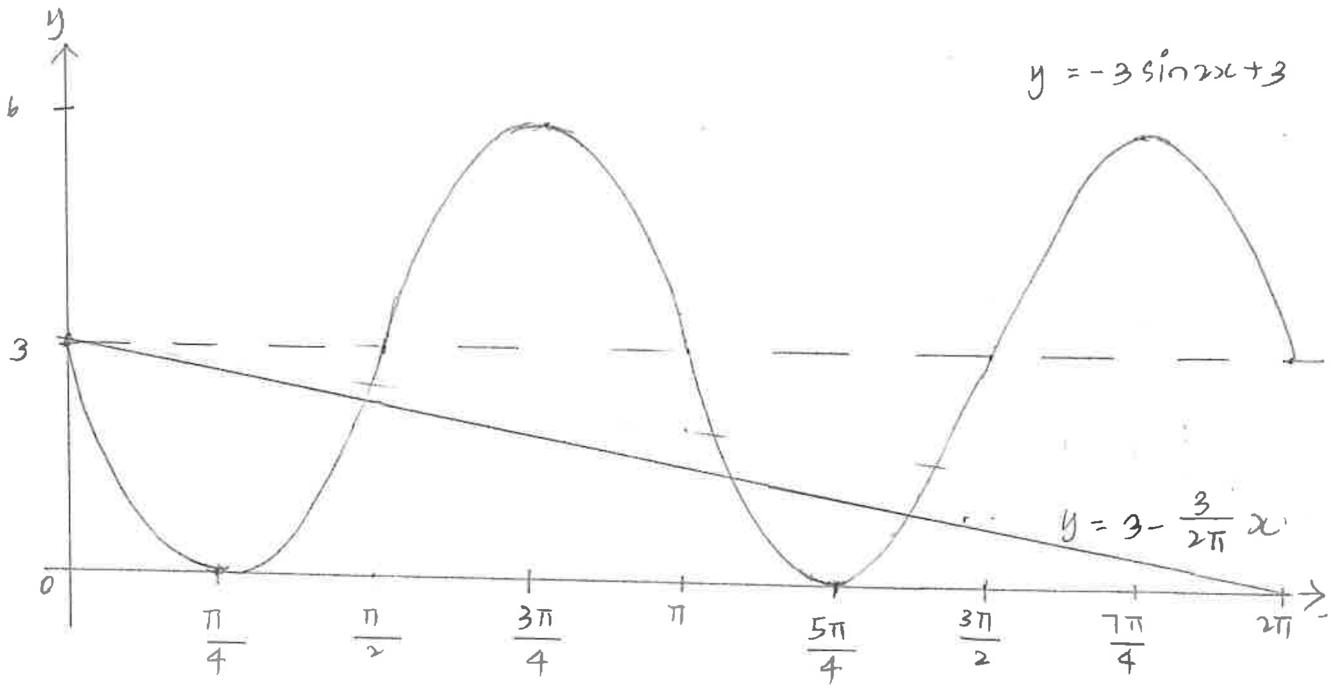
$$\frac{2\pi}{b} = \pi$$

$$b = 2$$

$$a = -3$$

$\therefore a = -3, b = 2, c = 3.$

4(b)



(c)

$$-3\sin 2x = -\frac{3x}{2\pi}$$

$$-3\sin 2x + 3 = 3 - \frac{3x}{2\pi}$$

$$y = 3 - \frac{3}{2\pi}x$$

Insert the line $y = 3 - \frac{3}{2\pi}x$.

Since there are 4 points of intersection,
$\therefore$ there are 4 solutions for the equation

$$a\sin bx = -\frac{3x}{2\pi}$$

5(a)

Given PA bisects $\angle QPB$,

$\therefore \angle APB = \angle APQ$.

$\angle ABP = \angle APQ$ (alternate segment)

$\therefore \angle APB = \angle ABP$.

Since $\angle APB = \angle ABP$, $\therefore \triangle ABP$ is isosceles with $\angle APB$ and $\angle ABP$ the base angles. Hence, $AB = AP$.

(proven).

(b)

$\angle ACP = \angle ABP$ ($\angle$ in the same segment).

$\angle ACB = \angle APB$ ($\angle$ in the same segment)

Since $\angle ABP = \angle APB$,

$\therefore \angle ACP = \angle ACB$.

In $\triangle CDP$ and $\triangle CBA$,

$\angle DCP = \angle BCA$ (proven above).

$\angle CPD = \angle CAB$ ($\angle$ in the same segment).

[By AA similarity test, $\triangle CDP$ is similar to $\triangle CBA$]

Hence, $\dfrac{CD}{CB} = \dfrac{DP}{BA}$

$BA \times CD = CB \times DP$

Since $AB = AP$,

$PA \times CD = CB \times DP$ (shown)

6a

$$T = 20 - 40e^{kt}$$

$$-1.95 = 20 - 40e^{k(1)}$$

$$40e^{k} = 21.95$$

$$e^{k} = \frac{21.95}{40}$$

$$k = \ln\left(\frac{21.95}{40}\right)$$

$$= -0.60011 \text{ (5s.f)}$$

$$= -0.600 \text{ (3s.f)}$$

6(b).

$T = -20 + 12$

$= -8.$

$-8 = 20 - 40e^{\ln \frac{21.95}{40} t}$

$40e^{\ln \frac{21.95}{40} t} = 28$

$e^{\ln \frac{21.95}{40} t} = 0.7$

$\ln \frac{21.95}{40} t = \ln 0.7$

$t = 0.59434...$

$= 0.594$ (3sf).

The time taken is 0.594h.

6c

since $e^{kt} > 0$ for $t \geq 0$

$-40e^{kt} < 0$

$20 - 40e^{kt} < 20$

$\therefore$ the temperature of the chicken can never reach $20°C$.

or consider $20 - 40e^{kt} = 20$

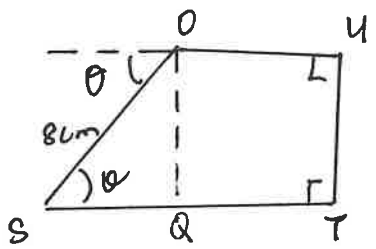
$40e^{kt} = 0$

$e^{kt} = 0$

Since $e^{kt} > 0$ for $t \geq 0$,
there is no t such that $e^{kt} = 0$.

$\therefore$ The temperature of the chicken can never reach $20°C$.

7(a)



$\angle OSQ = \theta$ (alt. $\angle$s, $OU \parallel ST$).

$\sin\theta = \frac{OP}{8}$

$OQ = 8\sin\theta$

$UT = OP$

$= 8\sin\theta$

$\cos\theta = \frac{SP}{8}$

$SQ = 8\cos\theta$

Perimeter of trapezium, P,

$= OU + OS + ST + TU$

$= 8\cos\theta + 8 + 2(8\cos\theta) + 8\sin\theta$

$= 8 + 24\cos\theta + 8\sin\theta$ (shown)

7(b)

$$P = 8\sin\theta + 24\cos\theta + 8$$

$$R = \sqrt{8^2 + 24^2}$$

$$= \sqrt{640}$$

$$= 8\sqrt{10}$$

$$\tan\alpha = \frac{24}{8}$$

$$\alpha = 1.24905$$

$$\alpha = 1.25 \ (3 \text{ sf})$$

$$P = 8\sqrt{10}\sin(\theta + 1.25) + 8$$

maximum possible perimeter

$$= 8\sqrt{10}(1) + 8$$

$$= 8 + 8\sqrt{10} \text{ cm}$$

[or 33.3 cm (3 sf)].

$$\theta + 1.24905 = \frac{\pi}{2}$$

$$\theta = 0.322 \ (3 \text{ sf})$$

corresponding $\theta = 0.322$

8a initial velocity

$t = 0$

$v = 4\cos^2(0) - 3$

$= 1$

The initial velocity is 1m/s.

8b $a = \frac{dv}{dt}$

$= 4\left[2\cos\left(\frac{t}{2}\right) \times \frac{1}{2} \times \left(-\sin\left(\frac{t}{2}\right)\right)\right]$

$= -4\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)$

$a = -2\,m/s^2$

$-2 = -4\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)$

$\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \frac{1}{2}$

$2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = 1$

$\sin 2\left(\frac{t}{2}\right) = 1$

$\sin(t) = 1$

$t = \frac{\pi}{2}$

The acceleration first reaches $-2\,m/s^2$ at $\frac{\pi}{2}$ seconds.

8c

$$S = \int v \, dt$$

$$= \int 4\cos^2\left(\frac{t}{2}\right) - 3 \, dt$$

$$= \int 4\left[\frac{\cos 2\left(\frac{t}{2}\right)+1}{2}\right] - 3 \, dt$$

$$= \int 2\cos t + 2 - 3 \, dt$$

$$= \int 2\cos t - 1 \, dt$$

$$= 2\sin t - t + c$$

$t=0$, $s=0$, $c=0$

$$s = 2\sin t - t$$

$v=0$

$$4\cos^2\frac{t}{2} - 3 = 0$$

$$\cos^2\frac{t}{2} = \frac{3}{4}$$

$$\frac{t}{2} = \cos^{-1}\sqrt{\frac{3}{4}}$$

$$= 0.52355$$

$$t = 1.0471 \text{ sec}$$

$t = 1.0471$

$s = 2\sin(1.0471) - (1.0471)$

$= 0.68485$ m. (5 sf)

$t = 4$

$s = 2\sin(4) - 4$

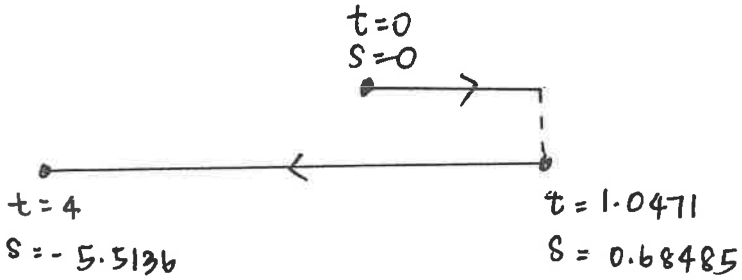
$= -5.5136$ m (5 sf).

Total distance travelled

$= 2(0.68485) + 5.5136$

$= 6.8833$ m (5 sf)

$= 6.88$ m (3 s.f)



9a

$$\pi r^2 h + \frac{2}{3}\pi r^3 = 200\pi$$

$$\pi(2x)^2(y) + \frac{2}{3}\pi(4x)^3 = 200\pi$$

$$4\pi x^2 y + \frac{128}{3}\pi x^3 = 200\pi$$

$$4\pi x^2 y = 200\pi - \frac{128}{3}\pi x^3$$

$$y = \frac{200\pi - \frac{128}{3}\pi x^3}{4\pi x^2}$$

$$= \frac{50}{x^2} - \frac{32x}{3}$$

9b

Total surface area, A

see * or

$$= \left[\pi(4x)^2 - \pi(2x)^2 + 2\pi(4x)^2\right] + \left\{\pi(2x)^2 + 2\pi(2x)(y)\right\}$$

$$= 16\pi x^2 - 4\pi x^2 + 32\pi x^2 + 4\pi x^2 + 4\pi x\left(\frac{50}{x^2} - \frac{32x}{3}\right)$$

$$= 48\pi x^2 + \frac{200\pi}{x} - \frac{128\pi x^2}{3}$$

$$= \frac{200\pi}{x} + \frac{16\pi x^2}{3}$$

$$= 40\pi\left(\frac{5}{x} + \frac{2x^2}{15}\right) \quad \text{(shown)}$$

*or $\pi(4x)^2 + 2\pi(4x)^2 + 2\pi(2x)(y)$

9c

$$A = 40\pi\left(\frac{5}{x} + \frac{2x^2}{15}\right)$$

$$\frac{dA}{dx} = \frac{-200\pi}{x^2} + \frac{32\pi x}{3}$$

For stationary point A, $\frac{dA}{dx} = 0$

$$\frac{-200\pi}{x^2} + \frac{32\pi x}{3} = 0$$

$$\frac{200\pi}{x^2} = \frac{32\pi x}{3}$$

$$600\pi = 32\pi x^3$$

$$x^3 = \frac{600\pi}{32\pi}$$

$$x^3 = \frac{75}{4}$$

$$x = \sqrt[3]{\frac{75}{4}}$$

$$= 2.6566 \quad (5s.f)$$

$\therefore$ Stationary value of A

$$= 40\pi\left[\frac{5}{2.6566} + \frac{2(2.6566)^2}{15}\right]$$

$$= 354.76 \quad (5s.f)$$

$$= 355 \quad (3s.f)$$

9d $\frac{d^2A}{dx^2} = \frac{400\pi}{x^3} + \frac{32\pi}{3}$

At $x = 2.6566$, $\frac{d^2A}{dx^2} = \frac{400\pi}{2.6566^3} + \frac{32\pi}{3}$

$= 101$ (3sf)

Since $\frac{d^2A}{dx^2} > 0$,

$\therefore$ the amount of paint needed is minimum.

10(a)

$$y = P - Q\cos 4x - \frac{1}{2}\sin 2x$$

$$\frac{dy}{dx} = 4Q\sin 4x - \cos 2x$$

$$\frac{d^2y}{dx^2} = 16Q\cos 4x - (-2\sin 2x)$$

$$= 16Q\cos 4x + 2\sin 2x$$

$$\therefore 16Q\cos 4x + 2\sin 2x + 4\left(P - Q\cos 4x - \frac{1}{2}\sin 2x\right) = 15\cos 4x - 8$$

$$16Q\cos 4x + 2\sin 2x + 4P - 4Q\cos 4x - 2\sin 2x = 15\cos 4x - 8$$

$$12Q\cos 4x + 4P = 15\cos 4x - 8$$

By comparison,

$$12Q = 15$$

$$Q = \frac{5}{4}$$

$$4P = -8$$

$$P = -2$$

$$\therefore P = -2, \ Q = \frac{5}{4}$$

10(b) At $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = 4\left(\frac{5}{4}\right)4\sin\left(\frac{\pi}{2}\right) - \cos 2\left(\frac{\pi}{2}\right)$$
$$= 1$$

Gradient of normal $= -1 \div 1$
$= -1$

At $x = \frac{\pi}{2}$, $y = -2 - \frac{5}{4}\cos 4\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right)$
$= -\frac{13}{4}$

$\therefore$ Equation of normal :

$$y - \left(-\frac{13}{4}\right) = -1\left(x - \frac{\pi}{2}\right)$$
$$y + \frac{13}{4} = -x + \frac{\pi}{2}$$
$$y = -x + \frac{\pi}{2} - \frac{13}{4}$$
$$y = -x + \frac{2\pi - 13}{4}$$

or Sub. $\left(\frac{\pi}{2}, -\frac{13}{4}\right)$ and $m = -1$ into $y = mx + c$

$$-\frac{13}{4} = -1\left(\frac{\pi}{2}\right) + c$$
$$c = \frac{\pi}{2} - \frac{13}{4}$$

$\therefore$ EQ. of normal : $y = -x + \frac{\pi}{2} - \frac{13}{4}$

$$y = -x + \frac{2\pi - 13}{4}$$