Tutorial 4: Functions

Section A (Basic Questions)

1 Find the range of the following functions:

(a) $\mathrm{f}: x \mapsto x^2 - 2x - 3, \quad x \in \mathbb{R}, \ -1 \le x \le 5$,

(b) $\mathrm{g}: x \mapsto x(x+5)(x-8), \ x \in \mathbb{R}, \ -3 \le x \le 5$,

(c) $\mathrm{h}: x \mapsto \tan x, \quad x \in \mathbb{R}, -\dfrac{\pi}{6} \le x \le \dfrac{2\pi}{3}, x \ne \dfrac{\pi}{2}$

[(a) $[-4,12]$ **(b)** $[-151,66.5]$ **(c)** $\left(-\infty,-\sqrt{3}\right] \cup \left[-\dfrac{1}{\sqrt{3}},\infty\right)$**]**

2 For each of the following functions, sketch the graph and determine if the inverse function exists. If it does, find it in a similar form.

(a) $\mathrm{f}(x) = x^2 - 4x + 3, \ x \in \mathbb{R}, \ x \le 1$, **(b)** $\mathrm{g}(x) = x^{\frac{2}{3}}, \quad x \in \mathbb{R}$,

(c) $\mathrm{h}(x) = \ln(2x-1), \ x \in \mathbb{R}, \ x > \dfrac{1}{2}$.

[(a) $\mathrm{f}^{-1}(x) = 2 - \sqrt{x+1}, \ x \ge 0$ **(c)** $\mathrm{h}^{-1}(x) = \dfrac{1}{2}\left(\mathrm{e}^x + 1\right), \quad x \in \mathbb{R}$ **]**

3 The function f is defined by $\mathrm{f}: x \mapsto 2-(x+1)^2, \ x \in \mathbb{R}, \ x \le k$.
Determine the largest value of k for which the function f^{-1} exists.

[largest value of $k = -1$**]**

Section B (Discussion Questions)

1 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} x^2 - 3 & \text{for } x \ \ge \ 2, \\ x - 1 & \text{for } x \ < \ 2. \end{cases}$$

Sketch the graph of f and find its range. Hence show that f is one-one and find f^{-1}.

$$\left[\mathrm{f}^{-1}(x) = \begin{cases} x+1 & \text{for } x < 1 \\ \sqrt{x+3} & \text{for } x \ge 1 \end{cases} \right]$$

2 Functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto -|x|, \ x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto \mathrm{e}^x \ , \ x \in \mathbb{R}..$$

Show that the composite function gf exists and find its range. **[** $(0,1]$ **]**

3 The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto 7(x-a)^2 - 1,\ x \in \mathbb{R},\ x < 3$$

$$\mathrm{g}: x \mapsto 1 + b - \mathrm{e}^{-x},\ x \geq 0,$$

where a and b are real constants.

(i) State the smallest value of a such that f^{-1} exists.

For the rest of this question, $a = 4$.

(ii) Explain if f^{-1} exists, and if it does, find $\mathrm{f}^{-1}(x)$ and state its domain.

(iii) Find the largest value of b such that the composite function fg exists. For this value of b, find $\mathrm{fg}(x)$, state its domain, and find its range.

[(i) $a = 3$ **(ii)** $\mathrm{f}^{-1}(x) = 4 - \sqrt{\dfrac{x+1}{7}}$, $\quad x > 6$

(iii) $b = 2$; $\mathrm{fg}(x) = 7(1+\mathrm{e}^{-x})^2 - 1$; $\mathrm{D_{fg}} = [0, \infty)$; $\mathrm{R_{fg}} = (6, 27]$ **]**

4 The functions f , g and h are defined as follows:

$$\mathrm{f}: x \mapsto (x-3)^3,\ x \geq 3,$$

$$\mathrm{g}: x \mapsto \ln(x+1),\ x > -1,$$

$$\mathrm{h}: x \mapsto \frac{1}{x},\ x > 0.$$

(i) Find $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1}, stating clearly their rules and domains. Hence state the set of values of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$.

(ii) The functions k_1 and k_2 are defined as follows:

$$\mathrm{k}_1: x \mapsto \ln\left(\frac{1}{x}+1\right),\ x > 0,$$

$$\mathrm{k}_2: x \mapsto x,\ \ x > 0.$$

Express k_1 and k_2 as appropriate compositions of the functions f, g and/or h.

[(i) $x \in [3, \infty)$ **]**

5 The function f is defined by $\mathrm{f}: x \mapsto 3 - (x-1)^2,\ x \leq 2.$

(i) Find the range of f. [1]

(ii) State, giving a reason, whether ff exists. [2]

The function g is defined by $\mathrm{g}: x \mapsto \mathrm{f}(x), \quad x \leq 1.$

(iii) Sketch the graphs of $y = \mathrm{g}(x)$, $y = \mathrm{g}^{-1}(x)$ and $y = \mathrm{g}^{-1}\mathrm{g}(x)$ on a single diagram, showing clearly their geometrical relationships. [4]

(iv) If $\mathrm{g}(\alpha) = \mathrm{g}^{-1}(\alpha)$, find the values of the constants p and q such that

$$\alpha^2 + p\alpha + q = 0.$$ [2]

(v) Find an expression for $\mathrm{g}^{-1}(x)$. [3]

[(i) $\mathrm{R_f} = (-\infty, 3]$ **(iv)** $p = -1,\ q = -2$ **(v)** $\mathrm{g}^{-1}(x) = 1 - \sqrt{3-x}$ **]**

6 Functions f and g are defined by

$$\mathrm{f}: x \mapsto |x+1|, \quad x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto \ln(x^2), \quad x \in \mathbb{R}, \quad x \neq 0.$$

(i) Sketch the graphs of f and g, and state the range of each function.

(ii) Explain why fg exists, define fg and state its range.

(iii) Using a graphical approach, solve the inequality $\mathrm{fg}(x) - \mathrm{f}(x) > 0$.

[**(i)** $\mathbb{R}_0^+$, $\mathbb{R}$ (ii) $\mathbb{R}_0^+$ (iii) $-5.36 < x < -0.703$ or $-0.464 < x < 0$ or $0 < x < 0.314$ **]**

7 The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \frac{1}{1-x}, \qquad x \in \mathbb{R}, x \neq 0 \text{ or } 1,$$

$$\mathrm{g}: x \mapsto 1 - \frac{1}{x}, \qquad x \in \mathbb{R}, x \neq 0 \text{ or } 1.$$

It is given that the range of g is $\mathbb{R} \setminus \{0,1\}$.

(i) Prove that the composite function fg exists. [2]

(ii) Express fg in a similar form and hence describe the relationship between f and g. [3]

(iii) Evaluate $\mathrm{f}^{2011}\mathrm{g}^{1994}\left(\frac{1}{2}\right)$. [3]

[(ii) fg : $x \mapsto x$, $x \in \mathbb{R}, x \neq 0$ or 1, f and g are inverse functions **(iii)** -1**]**

8 **(a)** The function f is given by $\mathrm{f}: x \mapsto 1 + \sqrt{x}$, for $x \in \mathbb{R}$, $x \geq 0$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Show that if $\mathrm{ff}(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence find the value of x for which $\mathrm{ff}(x) = x$. Explain why this value of x satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [5]

(b) The function g, with domain the set of non-negative integers, is given by

$$\mathrm{g}(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2 + \mathrm{g}(\frac{1}{2}n) & \text{for } n \text{ even}, \\ 1 + \mathrm{g}(n-1) & \text{for } n \text{ odd}. \end{cases}$$

(i) Find g(4), g(7) and g(12). [3]

(ii) Does g have an inverse? Justify your answer. [2]

[(a)(i) $\mathrm{f}^{-1}(x) = (x-1)^2, \mathrm{D}_{\mathrm{f}^{-1}} = [1, \infty)$ **(ii)** $x = \frac{1}{2}(3 + \sqrt{5})$ **(b)(i)** 6,8,9]

9 The functions f and g are defined as follows.

$$\mathrm{f}(x) = \sqrt{|2-x|}+1, \qquad x \in \mathbb{R},$$

$$\mathrm{g}(x) = \begin{cases} -\dfrac{1}{3}x + \dfrac{2}{3}, & 0 \le x < 2, \\ 1-(x-3)^2, & x \ge 2. \end{cases}$$

(i) Show that f^{-1} does not exist. [1]

(ii) If the domain of f is restricted to $[k, \infty)$ such that f^{-1} exists, state the least value of k and define f^{-1} in a similar form. [3]

Use the new domain of f found in part **(ii)** for the following parts.

(iii) Find the range of the composite function gf. [2]

(iv) Find the value of x such that $\mathrm{gf}(x) = 1$. [1]

[(ii) 2, $2 + (x-1)^2$, $x \ge 1$ **(iii)** $(-\infty, 1]$ **(iv)** 6**]**

10 It is given that

$$\mathrm{f}(x) = \begin{cases} (x-2)^2, & 0 \le x < 2 \\ 2x-4, & 2 \le x \le 4 \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x+4)$ for all real values of x.

(i) State the value of $\mathrm{f}(21)$. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-6 \le x \le 6$. [2]

The function g is defined by

$$\mathrm{g}(x) = \sqrt{x-4}, \quad 4 \le x \le 20.$$

(iii) Find fg in a similar form as f. [4]

$$\left[(\mathbf{i})\,1 \quad (\mathbf{iii})\ \mathrm{fg}(x) = \begin{cases} \left(\sqrt{x-4}-2\right)^2, & 4 \le x < 8 \\ 2\sqrt{x-4}-4, & 8 \le x \le 20 \end{cases} \right]$$